Volume XI. *November–December, 1900.* *Number 5.*

THE

PHYSICAL REVIEW.

EFFECT UPON THE PERSISTENCE OF VISION OF EXPOSING THE EYE TO LIGHT OF VARIOUS WAVE LENGTHS.[1]

By Frank Allen.

IN the early part of this century Plateau[2] engaged in some researches upon the persistence of vision. A disc composed of alternate black and white sectors was rotated at a speed just sufficient to give the surface the appearance of a uniform illumination, the velocity then giving the measure of the duration on the retina of the images of the white portions. By replacing the white sectors in turn with others of red, yellow and blue, he found that the duration of the images was different for the different colors, being least for the yellow and greatest for the blue. In a much more recent paper[3] the same writer extended his investigations, but used no method capable of giving more accurate or more comprehensive results.

Shortly after the publication of this latter paper, E. L. Nichols[4] introduced the method of using the spectrum instead of pigmentary colors. In the path of the ray of light producing the spectrum was placed a disc with alternate sectors cut out, the rotation

[1] Read before the American Association, New York, June 27, 1900.

[2] Plateau, Dissertation sur quelques propriétés des impressions produits par la lumière sur l'organe de la vue. Liege, 1829.

[3] Plateau, Sur une loi de la persistance des impressions dans l'oeil, Bruxelles, 1878.

[4] Nichols, On the duration of color impressions upon the retina. Am. Jour. Sci., Vol. 28, 1884.

of which gave flashes of light followed by equal periods of darkness causing a flickering of the portion of the spectrum in the field of view. The measurement of the velocity of rotation of the disc, at the point when the rapid flickering gave way to a continuous band of color, determined the duration of the maximum intensity of the sensation. A series of observations throughout the spectrum showed that the duration of the color sensation was least in the neighborhood of the *D* line, and increased towards both ends of the spectrum, the duration at the extreme violet being much greater than at the extreme red.

Taking advantage of this method, Ferry[1] made a more extended study of the persistence of vision, in which he increased the accuracy of the measurements by using an electric motor to rotate the disc the speed of which was recorded electrically on paper placed on the cylinder of a chronograph. With this apparatus Ferry more accurately investigated the law of the duration of retinal impressions, showing that it was a function of the luminosity and entirely independent of the wave length. In addition to this he obtained the persistency curves for a number of color blind persons. Comparison of their curves with that obtained for a perfectly normal eye disclosed the fact that in the regions to which the eye was color blind, the values of the persistence were greater than in the normal, while not differing in other parts of the spectrum.

In his investigations, Abney[2] has shown that the luminosity and persistency[3] curves of persons naturally color blind, or made so by disease, or by continued use of narcotics, invariably differed from similar curves for the normal eye.

Very recently Burch[4] has studied the spectrum by producing temporary blindness to the different colors in turn. Sunlight was allowed to stream through glass of the desired color and focused on the eye by means of a lens. After exposure for some minutes to this intensely bright color, the retina becomes irresponsive to that

[1] E. S Ferry, The Persistence of Vision. Am. Jour. Sci., Vol. 44, 1892.

[2] Capt. Abney, Colour Vision. Tyndall lectures at Roy. Inst., 1894.

[3] By "persistence" Abney means the reciprocal of the intensity of the light at the point of extinction, which is different from the meaning of the word in this paper.

[4] G. J. Burch, On artificial temporary color blindness. Proc. Roy. Soc., Vol. 63, 1898, p. 35. (Abstract.)

particular sensation, remaining so for a length of time depending on the wave length of the fatiguing color. The recovery of the retina was most rapid from the red and least from the violet. By this method, Burch confirmed the fundamental color sensations of the Young-Helmholtz theory, but in addition claimed evidence for believing blue to be a fourth sensation.

It was suggested to the writer that a combination of this method with that of Ferry would probably lend itself to a more conclusive determination of the fundamental color sensations.

Description of Apparatus and Method of Observation.

The apparatus consisted of a Browning two-prism spectrometer, a small motor with a disc attached, and an arrangement for registering the speed of the latter. An acetylene flame in a "Perfection" burner was the source of light, which gave a brilliant spectrum of considerable dispersion. The gas was supplied to the burner at a constant pressure. As the rotation of the disc caused air currents which disturbed the flame, it was found necessary to place the burner on a shelf attached to the wall at some distance behind the motor, and to focus the light on the slit by means of a lens. No flicker could be seen in the spectrum except that due to the motion of the disc, which rotated in the path of the rays of light between the lens and the slit. As the disc had alternate sectors of 90 de-

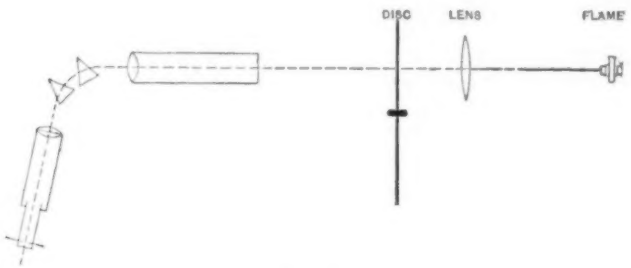

Fig. 1.

grees cut out, the duration of a single flash of light was therefore one quarter of the time of a complete revolution. The general diagram of the apparatus is shown in figure 1. The speed of the motor was regulated by means of a simple form of brake (see fig-

ure 2), made by passing a strip of leather over the smooth groove in the thumb screw that held the disc on the axle of the motor. One end of this leather was fastened to a spiral spring screwed to the table; the other had a long cord attached which was wound

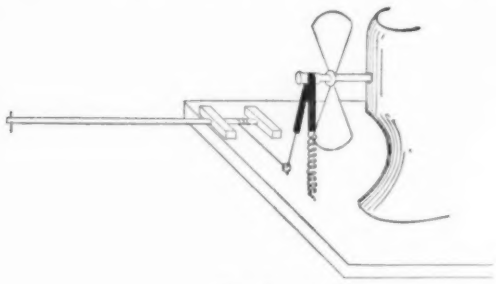

Fig. 2.

round a long brass rod turning in a friction socket, also screwed to the table. By turning this rod the string could be wound or unwound, and the friction of the leather on the axle of the motor increased or decreased. With this simple friction brake, the motor could be maintained constant at any speed from zero up to 1800 revolutions per minute for a period of several hundred revolutions.

To record automatically the speed of the disc a small hand speed-counter was clamped in a position so that its rubber tip

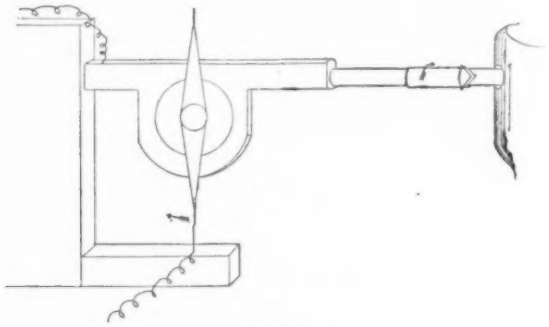

Fig. 3.

pressed firmly against the rear end of the axle of the motor (fig. 3). The rubber tip served to prevent the relative slipping of motor and speed-counter, and to insulate the latter, which was necessary as it carried a current. The small index on the counter was re-

placed by a long diametral arm, the ends of which in turn closed the primary circuit of an induction coil at every fiftieth revolution of the motor. The secondary circuit of this coil passed through metal parts of a chronograph, with a narrow spark gap at the surface of the cylinder. At every fiftieth revolution of the disc, therefore, a spark perforated a piece of paper carried on the cylinder. The pendulum of a clock, by a mercury contact, closed the primary circuit of another coil every second, and similar perforations were made at these intervals on the same piece of paper. The time of fifty or one hundred revolutions of the disc could then be easily measured, and the duration of the flashes of light on the retina computed. In order to have the registering apparatus completely and conveniently under the control of the experimenter, the primary circuits of both coils ran to a double mercury key within easy reach. At the proper moment both circuits were closed simultaneously, and the clock and motor automatically made their records for whatever length of time was necessary.

The method of taking observations was as follows: By means of a diaphragm in the eye-piece of the spectrometer a narrow portion of the spectrum was isolated. The motor was then started at such a speed that the flickering of the narrow band became quite distinct. By relaxing the friction-brake, the speed was gradually increased until the flickering just disappeared and a uniform band of color was presented to the eye. The speed of the motor was then kept constant while the circuits were closed and the clock and speed-counter made their records. Generally the time sufficient for one hundred revolutions of the disc was recorded as being most convenient for purposes of computation. A little practice enabled one to complete a setting in half a minute. Where greater accuracy was required, the slight tiring effect was avoided by making the first adjustments with the left eye, after which the setting could be obtained by the right eye with an exposure of less than half the time mentioned. After each observation in the case of the normal curves, the eye was rested in diffused daylight for at least ten minutes. Two and sometimes more settings were made at each part of the spectrum; and so constant were the different adjustments of the apparatus, the intensity of the light, and the sensitive-

ness of the eye, that observations taken at different times, and even on different days, differed from the mean value by less than two per cent.

The room within which the apparatus was set up was dark, and all extraneous light was excluded; the only illumination being due to such light as escaped through the screens surrounding the acetylene flame.

To reduce the circle readings of the spectrometer and thus identify the parts of the spectrum in which observations were made, readings were made for various Fraunhofer lines and a calibration curve of the usual form was plotted.

In this investigation a large part of the work done was with the eye fatigued with variously colored light. In order to conveniently attain this object, a plan was adopted differing from that described by Burch in the paper already cited. In the dark room an arc light was placed on a stand in front of which hung a dark cloth curtain. Light passing through a hole in this was focused by a lens upon the slit of a large direct vision spectroscope, which also was calibrated, so that the wave length of the fatiguing light could be found. A diaphragm in the eyepiece cut off all the brilliant arc spectrum except a narrow band about two millimetres in width of the required color. The intensity of the light also rendered a very narrow slit possible, thus increasing the purity of the spectrum. The eye was fatigued by allowing a narrow band of any pure color from this spectrum to fall upon the center of the retina for at least five minutes, a length of time found sufficient to give the degree of fatigue desired. This accessory apparatus was so placed that when the eye had become fatigued the observer, by simply turning in his chair, could make the setting for the persistence, as described above, without any loss of time, and before the eye had any time to recover to any appreciable extent from the tiring process. In taking a series of observations this fatiguing operation was repeated before each setting, after the completion of which the recording apparatus required readjustment, which gave the eye one minute rest. No extended rest was considered necessary during the progress of a series of measurements, as Burch found that any fundamental color sensation could be exhausted without affecting the others. That

this statement was abundantly justified will be seen upon a comparison of the normal persistency curves with those obtained for the eye when fatigued with various colors. The observations could therefore be taken with great rapidity.

When the eye was fatigued with any color, a piece of white paper placed where the white light from the arc fell upon it, appeared of a tint always complementary to the color affecting the eye. This afforded a convenient means of enabling the experimenter to determine the degree of exhaustion of the retina.

The time allowed for the fatiguing process with all colors was about five minutes. The uniformity of the measurements, for some of which this time was even doubled, seemed to indicate that with light of a given intensity the exhaustion of the color sensation reached a maximum in a few minutes, and that further exposure produced no effect. The following experiment served to confirm this inference. The eye was exposed to a band of pure red, the center of which had a wave length, $\lambda = .680\mu$, and persistence observations were made on a similar band of red the center of which had a wave length $\lambda = .758\mu$. Normal persistence $= .0179$ sec. The results are as follows:

Time of fatigue.	Persistence.
7 minutes.	.0195 sec.
5 "	.0193 "
10 "	.0188 "
20 "	.0188 "

Less than a minute rest was allowed between the readings except for the last which was made two days later. The eye accordingly was under a continued exhausting stimulus with the object of bringing about a greater degree of fatigue. The maximum exhaustion was apparently reached in seven minutes, though the other values may only differ by an experimental error. It may be, however, that the bright light tires the retina to a certain degree at first, after which the eye adjusts itself to the new conditions and partially recovers its normal condition. The above values would indicate this, though sufficient data are not yet available to establish this supposition. In any case the retina ultimately reaches a point of equilibrium between the destructive action of the external stimulus and the constructive power of the eye. If the luminosity of the source

of light is increased, the degree of exhaustion will undoubtedly be greater, and the maximum may be reached in the same or even in less time.

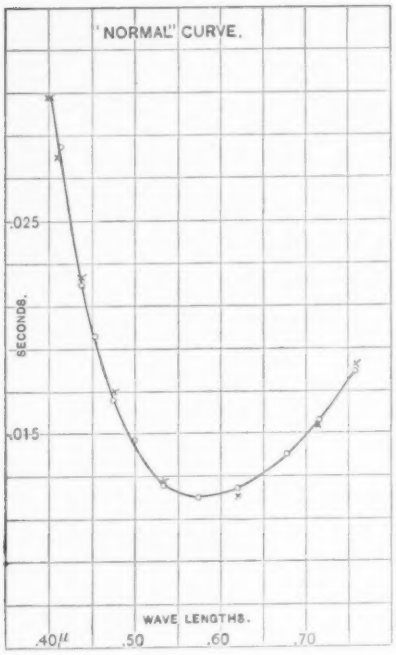

Fig. 4.

NORMAL PERSISTENCY CURVES.

The first persistency curve obtained was the normal. As this was the standard with which curves for abnormal conditions of the eye were to be compared, the readings were taken with great care to avoid all sources of error. The eye was allowed to rest in diffused daylight for more than ten minutes before each reading. The values of this series of settings are given in Table I. They are also graphically shown in Fig. 4, in which ordinates represent the persistence of vision for various parts of the spectrum in seconds, and abscissæ are wave lengths.

To guard against error, observations with the eye in its normal condition were frequently made to see if any change had taken

TABLE I.

Normal curve.

Wave length.	Persistence.	Wave length.	Persistence.
.400 μ	.0308 sec.	.533 μ	.0126 sec.
.415 "	.0285 "	.573 "	.0120 "
.438 "	.0220 "	.620 "	.0124 "
.454 "	.0196 "	.678 "	.0140 "
.475 "	.0166 "	.758 "	.0179 "
.500 "	.0147 "		

place. A series of such settings taken towards the latter part of the investigation is given in Table II. Their relation to corresponding readings upon the original normal curve is indicated by *X*'s in Fig.

5. This curve, like the ones given by Nichols and by Ferry in the papers cited, shows that the persistence is least near the *D* line and

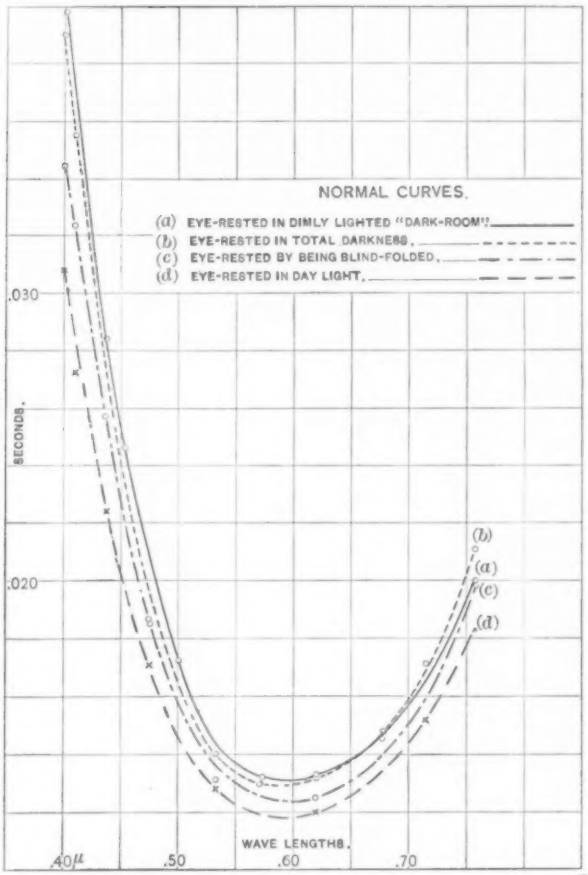

Fig. 5.

increases towards both ends of the spectrum. The curves are parabolic in form.

TABLE II.

Supplementary normal settings.

Wave length.	Persistence.	Wave length.	Persistence.
.400 μ	.0308 sec.	.533 μ	.0128 sec.
.410 "	.0272 "	.620 "	.0120 "
.438 "	.0224 "	.715 "	.0152 "
.475 "	.0171 "	.758 "	.0183 "

It was accidentally noticed at one time during a series of observations, that the duration of visual impressions was increased when the eye had been resting in the dark. This phenomenon was thereupon studied in some detail. Before each observation the eye was allowed to rest in the dark room in which the experiments were being conducted. This room was not completely dark, as the walls could be dimly seen, owing to the stray light from the acetylene

TABLE III.

Persistence after exposure to varying degrees of light and darkness compared with "normal" persistence.[1]

	Persistence under various Conditions.			
	a Eye rested in dark room.	*b* Eye rested in total darkness.	*c* Eye blindfolded.	*d*[2] Eye "normal" (exposed to diffuse day light).
.400 μ	.0398 △	.0390	.0344	.0308
.410 "	——	.0355	.0323	——
.415 "	——	——	——	.0285
.438 "	——	.0284	.0257	.0220
.454 "	.0246	——	——	.0196
.475 "	——	.0185	.0187	.0166
.500 "	.0172	——	——	.0147
.533 "	.0140	.0140	.0131	.0126
.573 "	.0132	.0130	——	.0120
.620 "	.0133	.0132	.0125	.0124
.678 "	.0145	.0148	——	.0140
.715 "	.0171	——	——	——
.758 "	.0200	.0211 ▽	.0199 ▽	.0179 ▽

flame. The results are given in column *a* of Table III. After coming out into the light, two of these readings were repeated and the following values obtained:

Wave length.	Persistence.
.678 μ	.0132 sec.
.620 "	.0125 "

On comparing these with the values given in Table III (*a*) for the same wave lengths, the difference will be at once noticed.

It was thought that, as the walls of the dark room could be seen indistinctly, the eye was subject to a strain which might be really

[1] Arrows indicate the order in which settings were made.

[2] Repeated for convenience of comparison from table I.

fatiguing it, and so increase the persistence of the color impressions. The observations were accordingly repeated, this time with the precaution of resting the eye in total darkness. The results are given in column *b* of Table III.

These values are a little greater in the red than the corresponding ones in column *a*, but the two curves lie very close together, as will be seen by inspection of Fig. 5, in which these curves are compared. It is quite evident, therefore, that the increased sensitiveness of the retina in the dark increases the persistence of vision, which is just the opposite of the anticipated effect. The increase in the values of the persistence of different wave lengths is not uniform, being about 15 per cent. greater than the normal in the extreme red, 11 per cent. in the yellow, and 30 per cent. in the extreme violet. This result harmonizes with a statement of Parinaud,[1] who found that "the increase in the sensitiveness for light which is produced on a retina placed in the dark, is not equal for rays of different refrangibility." He further states that this is true for all parts of the retina except the yellow spot. As the observations described in this paper extended at the same time over the yellow spot and surrounding portions of the retina, it is not possible to state conclusively whether the above statement is confirmed or not. The persistence of vision being a function of the sensitiveness of the retina as well as of the intensity of the light stimulus, connects what is found in the curves just described with the observation of the French writer.

In these last experiments the eye was kept open while being rested, and it was decided to further test the supposition that the increased values of the duration of the impressions might be due to the strain of adaptation of the retina, or to some kind of muscular strain. For this purpose the right eye, the one used in all these experiments, was rested by being closed and blindfolded. All muscular strain at least was removed in this way, and the results, in column *c*, Table III, show that a different condition was induced in the eye.

These values, shown graphically with the others in Fig. 5, are

[1] M. H. Parinaud, De l' intensitè lumineuse des couleurs spectrales. Comp. Rendus, Vol. 99, 1884, p. 937.

intermediate between what may be termed the ordinary and extra-ordinary normal curves. In this comparison it will be seen that the hypothesis that the great increase in the duration of the color impressions is due to some kind of muscular strain is substantiated partially, since the elimination of that possible disturbing factor brought about a decrease in these values. There still remains a substantial increase which must be accounted for, the most obvious explanation being that it is due to the increased sensitiveness of the retina. This curve also shows that the increased sensitiveness of the retina is unequal for waves of different lengths, the increase above the normal being 9 per cent. in the red, 5 per cent. in the yellow and 12 per cent. in the violet. These results may be looked at from another point of view. According to the Hering theory, the sensation, or rather the lack of sensation, black, has the power of assimilation of the white-black visual substance. The eye in its normal condition is under the mild influence of daylight which would dissimilate this substance to a certain extent. Darkness, being the absence of any stimulus of the retina, permits the restoration of the white-black substance to its maximum sensitiveness. In case this is the explanation of the phenomenon, why should the restoration of this visual substance affect the various colors at all, and especially different colors in different degrees? Instead of the visual substances of the Hering theory being independent of each other, some connection must exist between them. Darkness cannot be held to assimilate simply the white-black substance, but also the red-green and yellow-blue. It would at least be supposed that if restoring the white-black substance did affect the others, that it would affect the red-green substance equally in both red and green, and similarly the yellow-blue substance. But the persistence of the red is increased 9 per cent., while the increase in the green is 13 per cent. The discrepancy is far greater in the other case, as the increase in the yellow is only 5 per cent., while in the blue it is 13 per cent. These facts can hardly be said to be in harmony with the theory in question. Similar objections can not be urged against the Young-Hemholtz theory, since it does not group the color sensations into pairs of opposites, nor does it limit the restoring power of perfect rest—darkness—to counteracting any specific stimulus of the retina.

We have found so far that the persistence of vision is a maximum for all colors when the eye is rested by being kept open in total darkness. The values fall to an intermediate position when the eye is rested by being blindfolded, and reach a minimum when the rest has been given in diffused daylight. It would be expected that a considerable degree of exhaustion of the retina would have an opposite effect to that we have been discussing. But the experiments now to be described show that the opposite is the case—that fatiguing the retina causes the duration of the color impressions to increase again.

Effect of Fatiguing the Retina with Various Colors.

Retina fatigued with red.

The eye was fatigued, as described in the first part of this paper, by looking at an isolated portion of a brilliant arc spectrum for about five minutes, the process being repeated before every observation. The condition of the retina with the red sensation partially exhausted, approximates that of the naturally red-blind person.

In this process a band of color about 2 mm. wide was used, the center of which in the case of red had a wave length $\lambda = .680\ \mu$.

The results of the measurements of the duration of the various color impressions upon the retina in this abnormal condition, are exhibited in Table IV, and graphically in Fig. 6, in comparison with the normal curve.

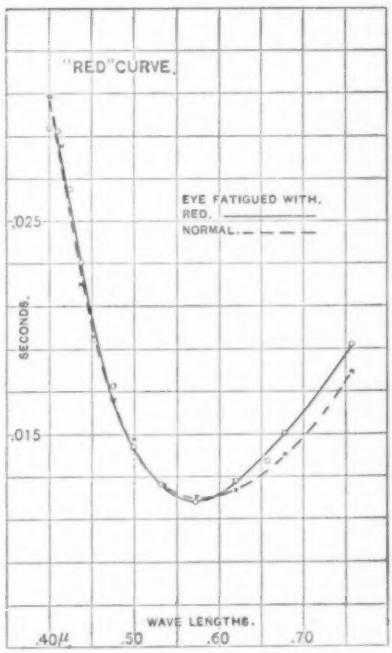

Fig. 6.

Comparing the two curves in Fig. 6, it is seen that fatiguing the eye with red has the effect of increasing the persistence of that color and also of about half of the orange, the remainder of the spectrum

being unaffected in any way. The persistence of the red is increased by 8 per cent., while the increase in the same portion of the spectrum when the eye was blindfolded was 9 per cent., the two agreeing within the limits of experimental error. The Hering theory would certainly lead one to suppose that the green part of the curve

TABLE IV.

Eye fatigued with red ($\lambda = .680\ \mu$.)

Wave length.	Persistence.	Wave length.	Persistence.
.400 μ	.0293 sec.	.533 μ.	.0126 sec.
.410 "	.0292 "	.573 "	.0118 "
.425 "	.0265 "	.620 "	.0128 "
.438 "	.0230 "	.657 "	.0137 "
.454 "	.0194 "	.678 "	.0150 "
.475 "	.0172 "	.758 "	.0192 "
.500 "	.0143 "		

would show some difference from the normal. A glance at the two curves, however, shows a perfect agreement between them. A similar lack of accordance of the experimental data with the anticipations of the Hering theory will be frequently noticed in the remainder of this paper.

It is to be noticed that the red sensation can be exhausted without affecting in any way the remaining sensations. This was a matter of some surprise; for upon looking at the acetylene spectrum after tiring the eye with red of the much more brilliant arc spectrum, all colors were apparently equally affected. The measurements showed that in this the eye was deceived. This confirms the statement of Burch in the paper previously cited, who, however, much more completely exhausted the color sensations than the author of this paper.

Retina fatigued with green.

The measurements in this case were made in the manner already described, but with the retina fatigued by a pure green light from the arc spectrum ($\lambda = .520\ \mu$.). The results are given in Table V and graphically in Fig. 7.

This curve shows an elevation or bunch in the green indicating, as before, an increase of the persistence in that part of the spectrum,

TABLE V.

Eye fatigued with green ($\lambda = .520\ \mu$).

Wave le gth.	Persistence.	Wave length.	Persistence.
.400 μ	.0311 sec.	.512 μ	.0150 sec.
.415 "	.0253 "	.533 "	.0126 "
.438 "	.0206 "	.573 "	.0120 "
.460 "	.0187 "	.620 "	.0125 "
.475 "	.0176 "	.678 "	.0137 "
.492 "	.0164 "	.758 "	.0185 "

due to the tiring process. The maximum increase of the persistence of the green is about 12 per cent. which again coincides with the increase in the persistence of the same wave length when the eye was blindfolded. The curve coincides with the normal in the yellow and red, but runs under it in the blue and violet. The Hering theory would lead us to expect the curves to differ in the red; the Young–Helmholtz in both violet and red if they differed at all. For if one sensation were exhausted it is possible that the intensity of the remaining sensations might be increased, producing the effect noticed in the blue and violet, and a similar effect in the red. The same phenomenon will be noticed in two other curves (see Figs. 8 and 9), though since it is but slight it may be due entirely to experimental errors.

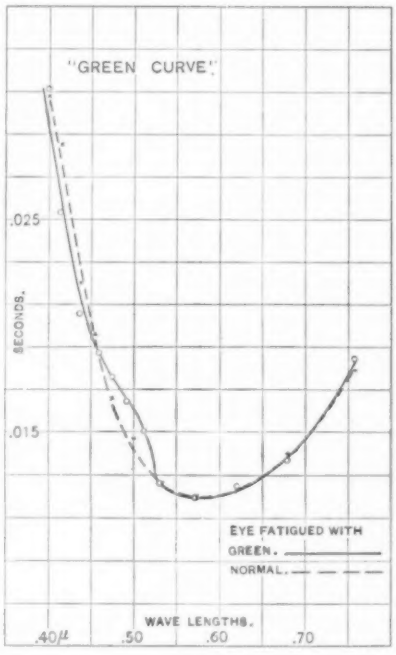

Fig. 7.

Retina fatigued with violet.

The following values were obtained when the eye was fatigued with violet:

TABLE VI.

Eye fatigued with violet ($\lambda = .400\ \mu$).

Wave length.	Persistence.	Wave length.	Persistence.
.403 μ	.0350 sec.	.502 μ	.0122 sec.
.410 "	.0287 "	.533 "	.0110 "
.423 "	.0227 "	.596 "	.0109 "
.434 "	.0198 "	.657 "	.0125 "
.443 "	.0188 "	.715 "	.0149 "
.454 "	.0161 "	.758 "	.0176 "
.468 "	.0149 "		

The data are plotted in Fig. 8 in comparison with the normal curve. The only part of the curve affected is the violet, which shows

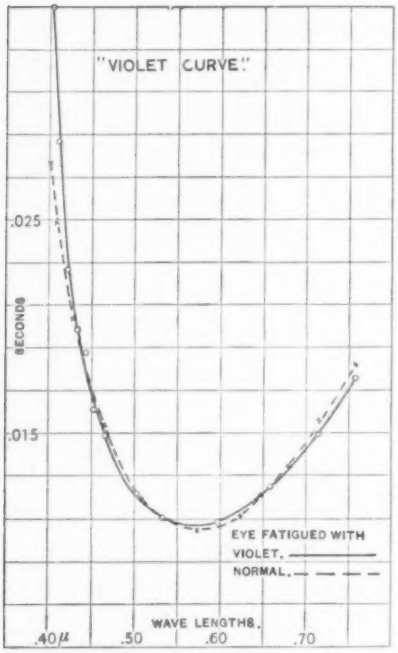

Fig. 8.

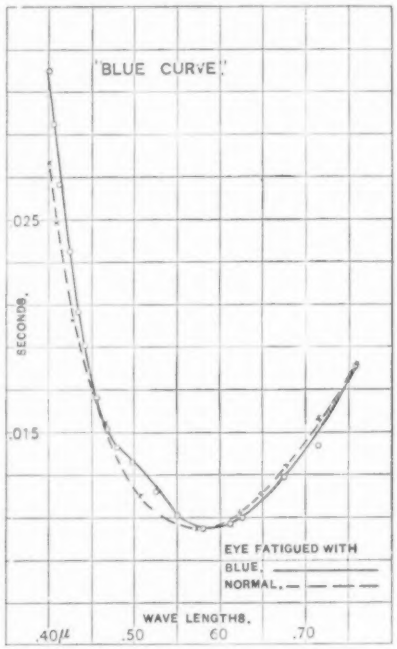

Fig. 9.

results similar to those obtained in the two preceding curves. There is an almost complete coincidence of the curves from the blue to the middle of the red, the fatigue curve then falling a little below the normal. This illustrates the remarks made upon a similar ef-

fect obtained in the abnormal curve for green. The persistence of the violet is increased by 27 per cent. These three curves for the eye tired with *red*, *green* and *violet* respectively, are of the same type, having one elevation which occurs in the part of the curve corresponding to the fatiguing color.

It would be quite natural to expect, in view of the results already discussed, that if blue and yellow are fundamental sensations in the same sense as are red and green according to either of the two most widely accepted color theories, similar elevations of the curve would be produced by using blue and yellow, respectively, as fatiguing stimuli, and these elevations would occur in the corresponding parts of the curves. In other words if blue and yellow are fundamental sensations, fatiguing the retina with these colors ought to affect these color sensations and no others. On the other hand, if blue and yellow are not fundamental, but compounded of two other sensations, then fatiguing with one of these ought not to show any effect in the part of the curve corresponding to that color, but should cause the elevations in the curve in the parts of the spectrum corresponding to those fundamental sensations which are affected. The latter is found to be the case.

Retina fatigued with blue.

Table VII shows the results obtained when the retina was exposed to blue light.

TABLE VII.

Retina fatigued with blue ($\lambda = .440\ \mu$).

Wave length.	Persistence.	Wave length.	Persistence.
.400 μ	.0320 sec.	.498 μ	.0136 sec.
.405 "	.0295 "	.525 "	.0122 "
.414 "	.0267 "	.550 "	.0111 "
.425 "	.0235 "	.580 "	.0105 "
.435 "	.0207 "	.612 "	.0107 "
.444 "	.0191 "	.625 "	.0110 "
.455 "	.0166 "	.675 "	.0129 "
.467 "	.0151 "	.715 "	.0143 "
.480 "	.0143 "	.758 "	.0182 "

These values will be found plotted in Fig. 9 in comparison with the normal curve. In the part corresponding to the blue, the two

curves coincide, showing that in this region the persistence has not been changed by the fatiguing process. The persistence has however been increased in both the green and violet, which is shown by the elevation of those parts of the curve. In this case the increase in the green is 10 per cent. while for the eye fatigued with green it was 12 per cent. The increase in the violet is 16 per cent. against 27 per cent. for the case when the eye was fatigued with violet. The readings in the extreme violet were, however, subject to a greater experimental error, as the illumination of that part of the spectrum was comparatively feeble. The curve obtained for the eye exhausted with blue, is therefore practically the superposition of the green and violet curves. The deduction is obviously that the sensation of blue is the consequence of the excitation of the sensations of green and violet. It will be noticed that the blue curve runs under the normal in the red, a phenomenon observed in two previous curves. This is shown best in the violet curve, where the abnormal runs under the normal in both green and red, indicating that when the retina has been exposed to violet this sensation is exhausted, but the other fundamental sensations red and green, are rendered a little more acute.

Retina fatigued with yellow.

We have now to examine the results obtained when the fatiguing color was yellow. The arc spectrum was dispensed with in this case, and the arc was replaced by a sodium flame. The yellow sodium light was of sufficient intensity to fatigue the eye which was placed at the focus of a lens. The following results were obtained:

TABLE VIII.

Retina fatigued with yellow ($\lambda = .589\ \mu$).

Wave length.	Persistence.	Wave length.	Persistence.
.400 μ	.0312 sec.	.533 μ	.0143 sec.
.415 "	.0302 "	.573 "	.0124 "
.438 "	.0236 "	.620 "	.0124 "
.454 "	.0205 "	.647 "	.0128 "
.475 "	.0177 "	.678 "	.0140 "
.492 "	.0163 "	.715 "	.0164 "
.512 "	.0155 "	.758 "	.0201 "

This curve, plotted in Fig. 10, in comparison with the normal curve, shows the yellow, blue and violet regions to be unaffected, while elevations were produced in the green and red. The whole curve is a little above the normal, which is probably due to a slight change in the pressure of the gas making the spectrum a little less luminous. The maximum increase in the persistence of red impressions in this case is 10 per cent., while in the case of exposure to red it was 8 per cent. This may be compared also with the increase in the red when the eye was blindfolded, which amounted to 9 per cent. In the yellow curve the increase in the green is 12 per cent. which is exactly the same as in the green curve. The position of the maximum in the two cases is however not the same.

Yellow light, therefore, has the same effect on the retina as red and green together, and it must be concluded that the sensation of yellow is due to a simultaneous disturbance of the red and green visual apparatus. Whatever may be the "deliverances of consciousness" as to the purity of blue and yellow, in the light of experiment we are forced to conclude that they are not fundamental but compound sensations, as far at least as their effect on the retina is concerned. Further than that, this paper does not pretend to deal with them.

Retina fatigued with white light.

Having successively fatigued the retina with five well-defined spectral colors with the interesting results described, some curiosity was evinced as to the probable effect of tiring the retina with all wave lengths at once, that is with white light. From the standpoint of the Young-Helmholtz theory, it was thought that the curve in this case should show three elevations; in the red, the green and the violet. On the other hand, assuming the truth of the Hering theory, the white light would affect the white-black visual substance which is independent of the other two. Color sensations affecting the like-wise independent red-green and yellow-blue substances should accordingly show no change in persistence. The result does not confirm either expectation. To accomplish the fatiguing of the retina, the white light from the arc passed through a lens and fell upon a piece of white paper placed at a little distance from the focus. The round spot of light was partly transmitted and fell

upon the retina for the usual time of five minutes. A series of after images followed which threatened to interfere with the accuracy of the observations. Upon looking at the portion of the spectrum in

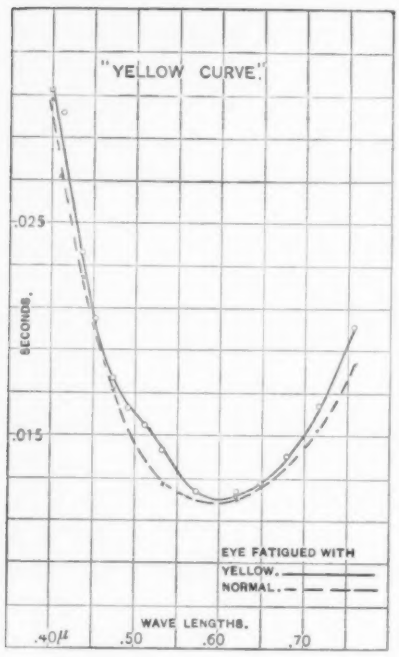

Fig. 10.

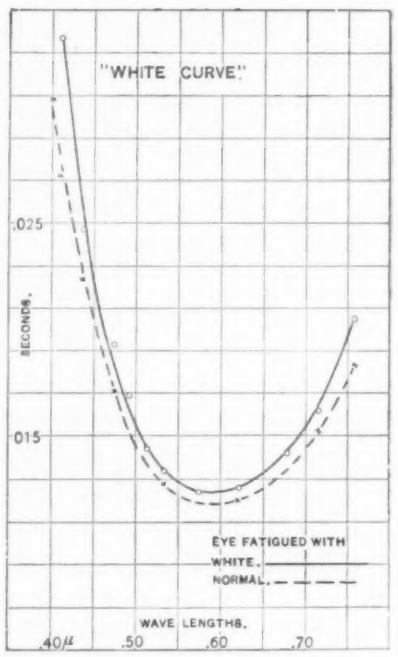

Fig. 11.

the field of view, it appeared as if seen through a hazy light of varying color. For the first few seconds this after image was of a faint yellowish tint which gradually deepened into an intense yellow completely obscuring the spectrum. The first hazy tint could be easily restored by winking, and this enabled the observer to take the readings always under the same conditions. The results given in Table IX, are so uniform as to inspire belief in their accuracy.

These data are plotted in figure 11 in comparison with the normal curve. The curves, it will be noticed, are approximately parallel, indicating no elevations in any part. Apparently this does not confirm the Young-Helmholtz theory of the sensation of white. The fact that the persistence of all the color sensations is increased shows an interdependence of the visual substances not provided for by the

Hering theory. But at any rate the curve seems to indicate the existence of a white sensation partially connected with and partially distinct from the fundamental color sensations. That white is either

TABLE IX.

Eye fatigued with white light.

Wave length.	Persistence.	Wave length.	Persistence.
.410 μ	.0337 sec.	.573 μ	.0124 sec.
.438 "	.0247 "	.620 "	.0126 "
.475 "	.0193 "	.678 "	.0142 "
.492 "	.0169 "	.715 "	.0161 "
.512 "	.0144 "	.758 "	.0205 "
.533 "	.0134 "		

an independent or semi-independent sensation, is claimed by Ebbinghaus and by Mrs. Franklin in their theories, and Abney[1] has recently obtained experimental results which point to the same conclusion.

In the discussion of the various curves thus far considered (Figures 4 to 10), the *percentage increase* of the duration of color impressions has been given. For greater convenience of comparison these values have been brought together in the following table :

TABLE X.

Percentage increase in the duration of color impressions when the eye is fatigued by exposure to various colors.

Region affected.	Color of the fatigue stimulus.						
	Red.	Green.	Violet.	Blue.	Yellow.	White.	Eye blindfolded.
Red.	8 %	0 %	0 %	0 %	10 %	9 %	12 %
Yellow.	0	0	0	0	0	5	5
Green.	0	12	0	10	12	13	7
Blue.	0	0	0	0	0	13	10
Violet.	0	0	27	16	0	12	14

When it is remembered that the values in this table represent experimental results obtained at intervals during several months, that no attempt whatever was made to keep the eye in a condition

[1] Abney, Sensitiveness of the Retina to Light and Color. Phil. Trans., A., 1897, Vol. 190.

of constant sensitiveness, and that the conditions under which the abnormal states of the retina were induced varied to a considerable extent from time to time, the results, especially in the red and green, show a uniformity that may very properly be considered remarkable. It can be no mere chance agreement of results. Certainly the retina, even under abnormal conditions, cannot have such a varying sensitiveness as is generally supposed. The values do not differ, for each color, by more than the presumable experimental error, except one value in the green and one in the violet, which will be evident on referring to the table. The averages of the tabulated results are as follows:

Red,	9.75	per cent.
Yellow,	5	"
Green,	10.8	"
Blue,	11.5	"
Violet,	17.2	"

In all these experiments the retina has apparently been brought to a maximum abnormal condition, though the fatiguing light has been of varying intensity. There are not sufficient experimental data yet to show whether greatly increasing the intensity of the light exhausting the retina, will increase this maximum condition or not, though it is anticipated that it will.

It has already been pointed out that, when the retina is made abnormally sensitive by being kept in the dark, the persistences of the various color sensations are unequally increased. The same is true when the retina is fatigued; for when any colors have their persistences affected, the increase is unequal in different parts of the spectrum, in all cases being least in the yellow and reaching maxima in the red and violet.

BEARING OF THE FOREGOING RESULTS UPON EXISTING THEORIES OF COLOR SENSATION.

The Hering Theory.

According to this theory, as is well known, the color sensations are arranged in three pairs, red-green, yellow-blue, and white-black. Since the first sensation in each pair is produced by the dissimilation of a supposed photo-chemical substance, which exists in three

forms to correspond to the three pairs of color processes, it would be expected that all being affected in precisely similar ways would show similar results. One would also expect the assimilating colors to show opposite effects. Whatever effect is produced by fatiguing the retina with red, an exactly similar effect should be shown with the yellow. So also for green and blue. Reference to the curves shows that red has an elevation in the red, while yellow gives two, neither of which corresponds to that color, and white gives none, or rather the whole curve is elevated above the normal. Similarly for the others—green, has one, blue two, and black none, if we may consider the curve obtained by blindfolding the eye as the one corresponding to black. We thus have three types of curves for the three pairs of color sensations. The agreement in number is the chief point of harmony with the Hering theory.

Analyzing the curves more particularly, we may observe, first, that acting on the red-green substance with red light has dissimilated it, producing a characteristic result. Assimilating this same substance with green light produces a similar result but in the part of the curve corresponding to green. Certainly the theory would expect each operation to affect both red and green parts of the curve, or else each would affect one and the same part but in different ways. Violet, too, giving a result similar in every way to those produced by red and green, must be placed on an equal footing with them, and whether it be considered as a dissimilation or an assimilation color its opposite must be found in the spectrum. Exhausting the retina with yellow discloses no special substance that is acted upon, but elevations are produced in both red and green. Translating this into the language of the theory, yellow simultaneously exercises the functions of both red and green to the same percentage amount, as we have seen, as each of them separately. Under these circumstances a balance should have been maintained between the opposing actions of red and green, which should have produced a curve differing in no respect from the normal. Supposing however that the action of the yellow stimulus is in harmony with the requirements of the theory, exposure of the retina to its assimilative companion blue ought to counteract the effects previously produced. Experiment shows that the green is affected, but

in addition the violet is also affected which was not disturbed at all by the yellow. Further, by stimulating the retina with strong yellow and blue lights alternately, the red-green substance could be overbalanced, and red-blindness induced by the action of certain luminous waves which according to theory should have no effect in this direction. It is quite evident, therefore, that these experimental results are antagonistic to the theory in question. The curves obtained when the eye was rested in the dark and fatigued with white light at first sight seem to confirm the Hering theory; but since in both cases all the color sensations are affected, there must be an intimate connection between white, black, and the color processes. This, too, is not in harmony with the provisions of the Hering theory.

The Ebbinghaus Theory.

The same objections will also apply to the theory of Ebbinghaus,[1] which is practically a modification of that of Hering. The main difference is that while the latter postulates three visual substances which may be regarded as so many independent variables, the former separates these substances into four for the perception of color and one for white, making in all five independent variables. The fundamental color tones in this theory are the same as in that of Hering, viz., red, yellow, green and blue, two of which, yellow and blue, have been shown to be compound.

The Franklin Theory.

The ingenious color theory of Mrs. Franklin[2] is framed partly from the physical and partly from the psychological standpoint. This evolutionary theory regards color-sense as a development out of an earlier existing gray-white sensation. Guided only by the requirements of consciousness the fundamental color tones would be red, green, yellow and blue. But physical experiments have rendered it improbable that yellow is fundamental, so the theory is resolved into a three-color one, with a gray-white sensation in addition. Consciousness must be satisfied with only a part of its demands.

[1] Ebbinghaus, Theorie des Farbensehens. Zeitschrift für Psychol. und Phys. der Sinnesorgane, Bd. V., Heft 3, 1893.

[2] C. L. Franklin, On Theories of Light-Sensation, Mind, Vol. 2, N. S. No. 8, 1893.

The experiments described in this paper are equally fatal to both blue and yellow as fundamental sensations, but can be considered hostile to this theory only as regards the blue. Mrs. Franklin might then go one step farther and assume the Young-Helmholtz color-sensations by the substitution of violet for blue.

The Young-Helmholtz Theory.

The experiments we have described are plainly seen to harmonize

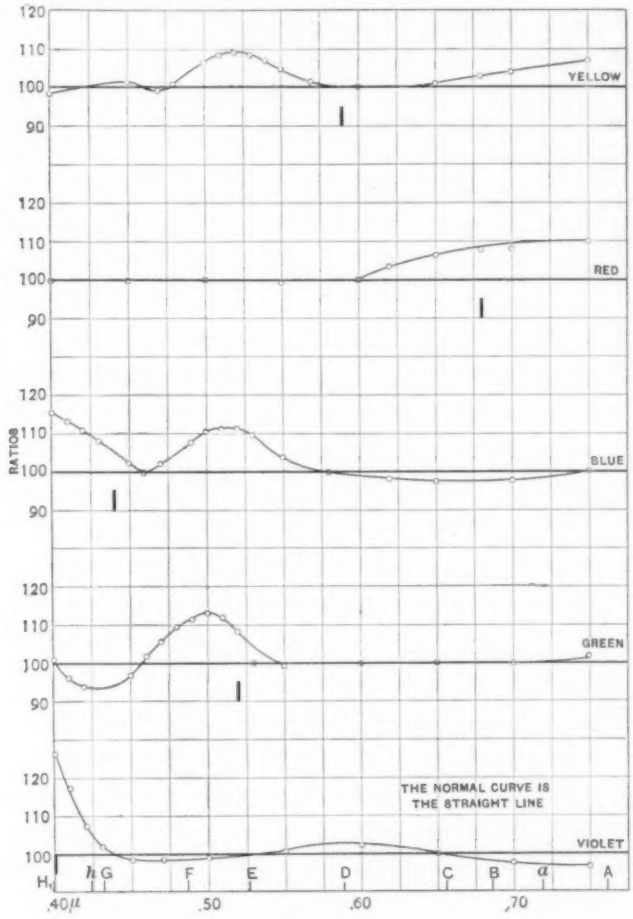

Fig. 12.

completely with the fundamental color processes of the Young-Helmholtz Theory. Red, green and violet being these fundamental sen-

sations give characteristic results in an abnormal condition. Yellow and blue are both compounded of two other sensations, and experiment shows the theory verified. Rest in the dark makes all sensations more sensitive, which is shown by the curve obtained under this condition. The curve obtained for the action of white light, however, cannot be disposed of so easily, and, as before, we must consider it as evidence in support of a white sensation. There are many other color theories; but these we have considered may be considered typical of all.

At the suggestion of Dr. Nichols these curves have been drawn in another way which shows the deviations of the abnormal from the normal curves in a much more striking manner. All parts of the normal curve are reduced to the same ordinate, making that curve a straight line parallel to the axis of abscissæ. The corresponding values of the abnormal curves are then reduced in the same ratio and these ratios plotted. For convenience the ordinate of the normal curve is taken as 100. The variations from the normal, discussed in the preceding pages, are much more evident especially in the parts corresponding to the blue and violet regions of the spectrum. It is evident that this is a representation of what would

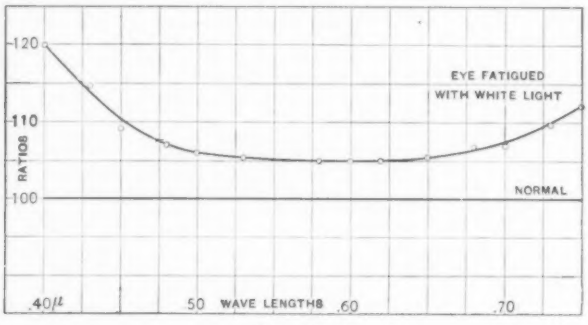

Fig. 13.

actually have been obtained, had all parts of the spectrum possessed the same luminosity.

Only those curves obtained when the eye was fatigued with various colors and with white light are so reduced, as the others do not show their characteristics any better by this method. The five color curves are plotted together for comparison in Fig. 12, the

white in Fig. 13. In addition the same curves are plotted in Fig. 14, with circle readings as abscissæ instead of wave lengths, which

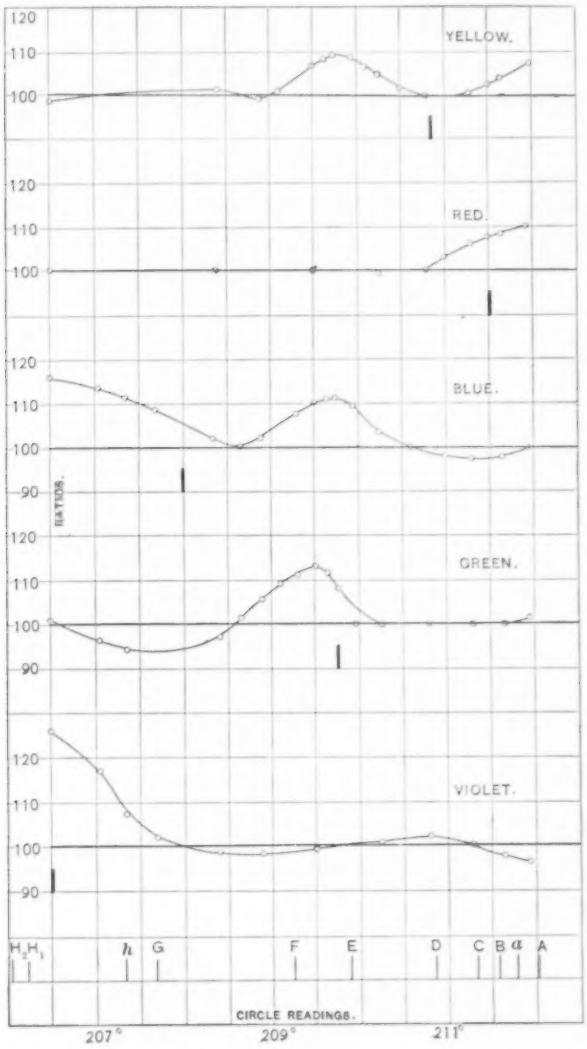

Fig. 14.

represents the conditions of experiment more accurately, as the observations were made on a prismatic spectrum.

In these last diagrams the part of the spectrum used for fatiguing

the eye is marked by means of a vertical line under the respective curves.

EXAMINATIONS OF NATURALLY COLOR-BLIND PERSONS.

S. W. F., a student in Cornell University who was known to be color-blind, kindly consented to allow his eyes to be tested. Being accustomed to make scientific measurements, his observations were made with care and accuracy. The limits of the spectrum were the same to him as to the normal eye. The apparatus did not permit his neutral point to be determined accurately, but he indicated it to be at about $\lambda = .480\ \mu$. As is usual with red-blind persons, he called all on the red side of the neutral point yellow, and the other blue. When tested with the Holmgren worsteds he made the usual matches of the red-blind. In addition, however, he indicated blindness of some other part of the retina, as shown by his matches of light yellows and greens. Measurements were made *for both eyes* in the manner described for obtaining normal curves.

The following table gives the persistence of color impressions for the right eye:

TABLE XI.

S. W. F. (right eye).

Wave length.	Persistence.	Wave length.	Persistence.
.410 μ	.0297 sec.	.533 μ	.0133 sec.
.438 "	.0246 "	.550 "	.0128 "
.455 "	.0214 "	.573 "	.0133 "
.465 "	.0202 "	.596 "	.0124 "
.475 "	.0184 "	.620 "	.0128 "
.484 "	.0170 "	.678 "	.0161 "
.493 "	.0162 "	.715 "	.0191 "
.512 "	.0148 "	.758 "	.0224 "

The data are plotted in Fig. 15 in comparison with the third normal curve of the writer. These values show a very great increase in the duration of the impressions in the red; the curve then approaches nearly to the normal in the yellow, then rises in the yellowish-green forming a curious little bunch or elevation, thence continuing parallel to the normal through the blue and violet. It will be noticed that the whole curve is displaced upward, which in-

dicates that the whole spectrum was not as bright to the color-blind eye as to the normal. This bunch in the yellow-green is not accidental nor is it due to an erroneous measurement, since it was confirmed by experiments several weeks apart. It is also confirmed in the curve for the left eye of the same subject, shown in figure 16 in

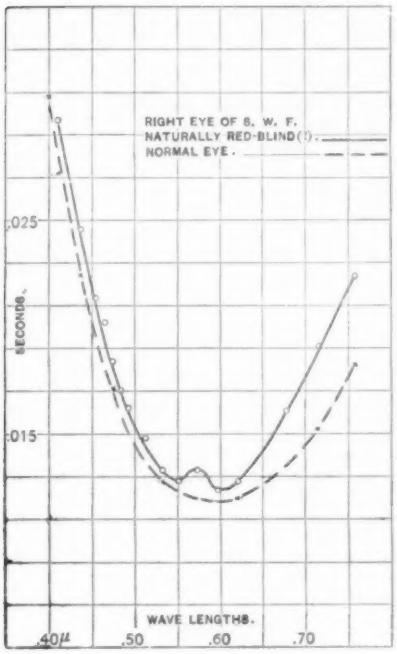

Fig. 15.

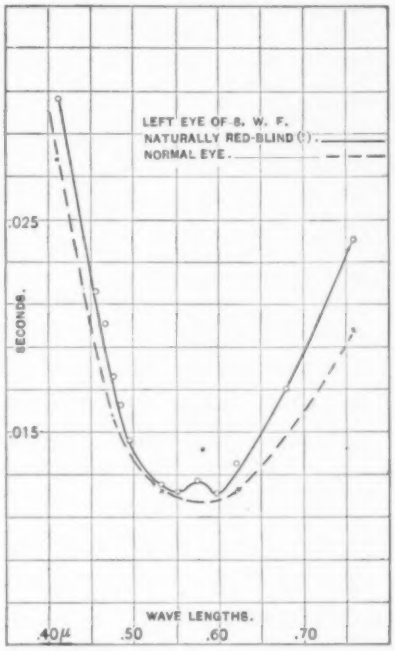

Fig. 16.

comparison with a new normal curve of the writer, the determination of which was rendered necessary by a change in the pressure of the acetylene and consequently a change in the luminosity of the spectrum.

The form of the curves is the same for both eyes, both showing the same peculiarities. It may be possible that this is the curve of the typical red-blind subject. Ferry, however, in the article cited, found curves slightly different from these for red-blind persons. In his curves there were no elevations in the yellow-green part of the spectrum. The curve for the normal eye (Fig. 6), fatigued with red, shows the increased persistence of the red though not to the

same extent as the naturally color-blind subject. No disturbance of the curve occurs in any other part.

Table XII gives the data for the left eye.

TABLE XII.

S. W. F. (left eye).

Wave length.	Persistence.	Wave length.	Persistence.
.410 μ	.0307 sec.	.550 μ	.0122 sec.
.455 "	.0216 "	.573 "	.0127 "
.465 "	.0201 "	.596 "	.0121 "
.475 "	.0176 "	.620 "	.0135 "
.484 "	.0162 "	.678 "	.0170 "
.493 "	.0146 "	.758 "	.0240 "
.533 "	.0125 "		

Mr. J. E. T., of the Chemistry department of the University, informed the writer that he had some difficulty in observing red lines in spectroscopic analysis, and kindly consented to make the necessary observations for his persistency curve. When tested with the Holmgren worsteds he matched the proper colors with the confusion samples, and seemed as far as that is concerned to have normal color vision. But finally being tested with other samples he made matches which indicated a partial red-blindness at least, and also a disturbance of the violet. He stated also that some difficulty was experienced with violet lines in spectroscopic work.

The values of the persistence are given in the following table:

TABLE XIII.

J. E. T.

Wave length.	Persistence.	Wave length.	Persistence.
.400 μ	.0428 sec.	.574 μ	.0112 sec.
.415 "	.0296 "	.596 "	.0114 "
.438 "	.0214 "	.620 "	.0119 "
.475 "	.0157 "	.678 "	.0144 "
.500 "	.0136 "	.715 "	.0184 "
.533 "	.0117 "	.758 "	.0224 "
.550 "	.0114 "		

These values, for the right eye, will be found in Fig. 17 in comparison with a normal curve of the writer. Both eyes, the right and

left, gave approximately the same curve, only one of which is here plotted. It will be noticed that the part of the curve corresponding to red shows a characteristic elevation above the normal, indicating red-blindness. Through the yellow-green the points on the curve were obtained quite close together to see if any elevation occurred as in the previous case. No deviation from a smooth curve was, however, obtained. The persistence of green is shown to be, in this case, less than in the normal eye, indicating that, to the former, green was relatively much brighter. This is quite in agreement with other cases of color-blindness studied by various observers. Comparison of the two curves also shows that the lowest point on

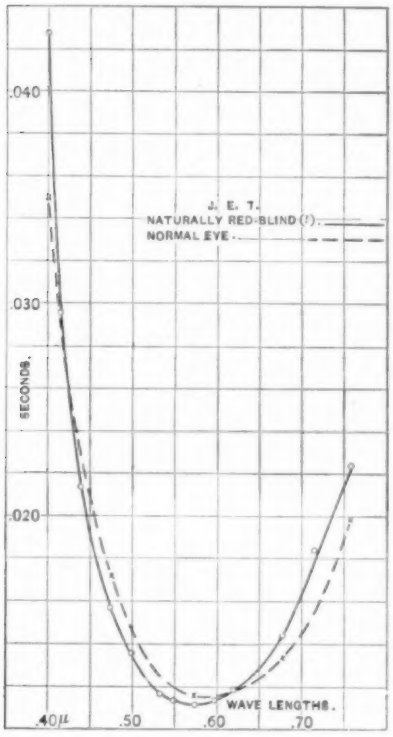

Fig. 17.

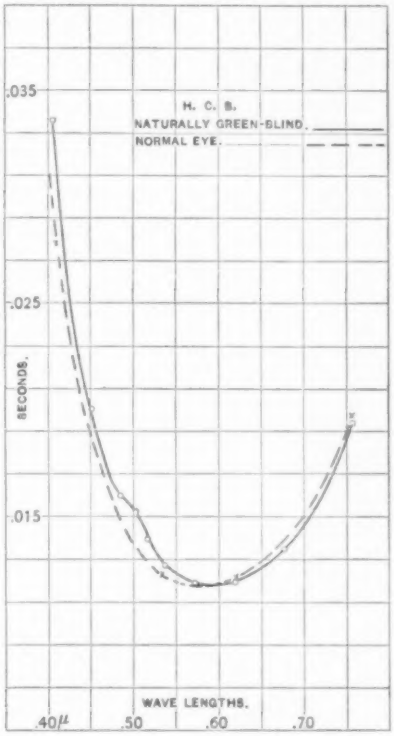

Fig. 18.

that obtained for the color-blind eye does not correspond to the same part of the spectrum as the similar point on the normal, but is displaced towards the blue end of the spectrum. This indicates

that the point of maximum brilliancy is shifted in that direction, a result also commonly observed. The curve also shows a persistence greater than normal in the extreme violet. Though there is greater difficulty in making observations in this region than in any other, yet since all Mr. T's readings here were greater than for the normal eye, it must indicate some abnormal condition of the retina as regards its appreciation of violet. It is not possible to say conclusively, without further investigation, whether this is a case of violet color blindness or not.

Another student of the University, Mr. H. C. B., also consented to determine his persistency curve. Upon examination with the worsteds he indicated green-blindness. Persistency curves for both eyes were obtained, but their close agreement renders only one necessary. The following values for the left eye were obtained:

TABLE XIV.

H. C. B. (left eye).

Wave length.	Persistence.	Wave length.	Persistence.
.405 μ	.0336 sec.	.538 μ	.0127 sec.
.454 "	.0201 "	.574 "	.0119 "
.487 "	.0160 "	.620 "	.0120 "
.502 "	.0152 "	.678 "	.0135 "
.517 "	.0139 "	.758 "	.0194 "

These values will be found plotted in Fig. 18 in comparison with a curve for the normal eye. The characteristic elevation in the green is well shown, though the persistence is not increased to quite the extent anticipated. It may indicate only a partial green-blindness. No other part of the curve differs from the normal. This curve, and the one preceding, show that an eye may be color-blind to red or green without necessarily affecting any other sensation, a conclusion which does not agree with the Hering theory of color blindness.

From these three curves for color blindness, it is not possible to draw many definite conclusions; but there is evidently reason to believe that the study of many cases of color blindness by this method will throw a new light upon this interesting subject, and

may lead to important evidence as to the real nature of color-vision itself. Preparations for such an investigation, on the part of the writer, are being made.

Summary.

The results arrived at in this investigation may be conveniently summed up as follows:

1. The persistence of all color impressions reaches a maximum when the eye has been kept open in the dark, the increased duration being unequal for rays of different refrangibilities. The increase is least in the yellow, and grows larger towards both the red and violet. The curve obtained under such conditions may be called the *extraordinary* normal curve.

2. The persistence of all color impressions is a minimum when the eye is under the influence of diffused daylight. The curve thus obtained has been termed in this paper the *ordinary* normal curve.

3. When the eye has been blindfolded the values of the persistence in all cases are intermediate between the extraordinary and ordinary normal curves, showing, however, unequal increases as in the extraordinary curve.

4. When the retina has been exposed to strong red, green or violet light, the persistence of impressions of the same color as the fatiguing light is increased, forming a characteristic elevation or bunch in the part of the persistency curve corresponding to that color.

5. When the retina has been fatigued with yellow or blue light there is no increase in the persistence of impressions of those colors, but the characteristic elevations are produced in the parts of the curves corresponding to red and green or green and violet respectively.

6. These phenomena strongly confirm the fundamental color sensations of the Young-Helmholtz theory.

7. When the retina has been fatigued with white light, the persistences of all color impressions are unequally increased. This apparently does not harmonize with the Young-Helmholtz theory, but indicates a partially independent white sensation.

8. Both rest and fatigue of the retina cause an increase in the

persistence of all colors. No explanation of this fact has yet been found.

9. In the case of naturally color-blind eyes, the persistence of color impressions to which the retina is blind is abnormally increased, thus affording a method of determining the wave lengths to which the retina is incapable of responding normally.

10. In addition to the loss of one fundamental color sense, the naturally color-blind retina may also lack the power of responding to a group of waves affecting another sensation. The further study of cases of color blindness in this way would probably throw new light upon this interesting and important subject.

In conclusion, I wish to gratefully acknowledge my indebtedness to Professor E. L. Nichols, who suggested the subject and method, and also to Mr. J. S. Shearer, of the department of Physics, for their many valuable suggestions regarding both apparatus and the experimental results of the entire investigation.

PHYSICAL LABORATORY,
CORNELL UNIVERSITY, May, 1900.

GIBBS THERMODYNAMICAL MODEL FOR A SUBSTANCE FOLLOWING VAN DER WAALS' EQUATION.

By W. P. Boynton.

VAN DER WAALS' equation is commonly written

$$(p + a/v^2)(v-b) = RT. \tag{1}$$

For the purposes of a general discussion it is often convenient to suppress the three constants a, b, R, by taking the critical pressure volume and temperature as the units of pressure, volume, and temperature respectively. If these be called p_c, v_c, T_c, and the so-called *reduced* pressure, volume, and temperature be indicated by

$$\pi = \frac{p}{p_c}, \quad \varphi = \frac{v}{v_c}, \quad \vartheta = \frac{T}{T_c}.$$

It can be shown that the equation will reduce to the form

$$\left(\pi + \frac{3}{\varphi^2}\right)(3\varphi - 1) = 8\vartheta, \tag{2}$$

which *appears* to have no empirical constants.

The form of the isothermals, with whose general properties we are familiar, is shown in Fig. 1. In this figure we notice certain peculiarities. These we will discuss a little later.

In connection with questions raised in a previous article[1] it seemed to me desirable to find the form of a Gibbs' Thermodynamical Model for a substance whose properties in two states were represented by a comparatively simple analytical expression. A substance following van der Waals' equation is as much an ideal substance as is a perfect gas, yet it exhibits many properties common to actual substances. For a substance following equation (1) the first law of Thermodynamics

[1] Phys. Rev., 10, p. 228, 1900.

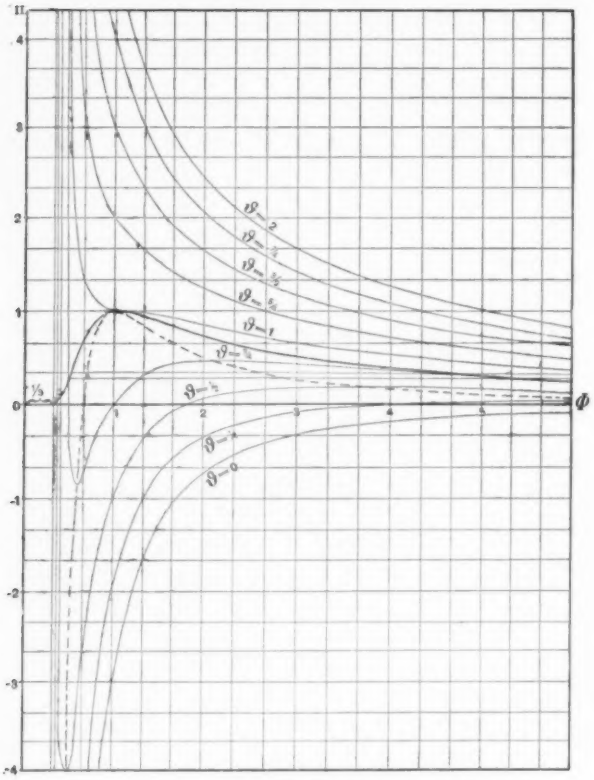

Fig. 1.

$$dQ = dU + pdv.$$

becomes

$$dQ = C_v dT + (p + a/v^2)dv \tag{3}$$

from which it readily appears that

$$dU = C_v dT + a/v^2 dv$$

and if we regard C_v as a constant, an assumption entirely in harmony with, and even suggested by, those ideas of the Kinetic Theory which gives us the original equation, we have

$$U = \int_o^T C_v dT + \int_b^v \frac{a}{v^2} dv = C_v T + \frac{a}{b} - \frac{a}{v}. \tag{4}$$

The coördinates of the surface we wish are the energy, volume and

entropy. We may introduce the latter and eliminate T by comparing (3) with

$$dQ = TdS$$

from which we can deduce

$$S = C_v \log T + R \log (v - b) + \text{Const} \tag{5}$$

$$T = K' e^{\frac{S}{C_v}} (v - b)^{-\frac{R}{C_v}}$$

and substituting in (4),

$$U = K e^{\frac{S}{C_v}} (v - b)^{-\frac{R}{C_v}} + a/b - a/v \tag{6}$$

where $$K = C_v K'.$$

R/C_v is a constant, equal, according to the Kinetic Theory, to two-thirds of the ratio of the kinetic energy of translation of the molecules to their total energy, or to $\frac{C_p}{C_v} - 1$ very nearly, that is, in ideal cases such as we are considering, to 2/3, 2/5, 2/6, 2/7, etc. This then is the equation of the model, and we see immediately that it satisfies the differential equations

$$T = \frac{\partial U}{\partial S} = K e^{\frac{S}{C_v}} (v - b)^{-\frac{R}{C}} / C_v = K' e^{\frac{S}{C_v}} (v - b)^{-\frac{R}{C_v}}$$

$$p = -\frac{\partial U}{\partial v} = \frac{R}{C_v} K e^{\frac{S}{C_v}} (v - b)^{-1-\frac{R}{C_v}} - \frac{a}{v^2}$$

$$= \frac{RT}{v - b} - \frac{a}{v^2}.$$

For a general discussion it is desirable to have the equation in terms of the reduced coördinates. At the critical state equation (5) becomes

$$S_c = C_v \log T_c + R \log (v_c - b) + \text{Const}$$

and hence

$$S - S_c = C_v \log \frac{T}{T_c} + R \log \frac{v - b}{v_c - b}$$

or remembering that

$$T/T_c = \vartheta, \quad b = v_c/3, \quad v/v_c = \varphi$$

(7) $$S - S_c = C_v \log \vartheta + R \log \frac{3\varphi - 1}{2}$$

whence

(8) $$\vartheta = e^{(S-S_c)/C_v} \left(\frac{3\varphi - 1}{2}\right)^{-R/C_v}$$

Equation (4) $$U = C_v T + a/b - a/v$$

gives, substituting for T and v their values

$$T = \vartheta T_c = \frac{8a}{27bR}\vartheta$$

$$v = \varphi v_c = 3b\varphi$$

$$U = \frac{8aC_v}{27bR}\vartheta + \frac{a}{b} - \frac{a}{3b\varphi}$$

$$= \frac{a}{b}\left(\frac{8C_v}{27R}\vartheta + \frac{3\varphi - 1}{3\varphi}\right)$$

and substituting the value of ϑ just found,

$$U = \frac{a}{b}\left[\frac{8C_v}{27R} e^{(S-S_c)/C_v} \left(\frac{3\varphi - 1}{2}\right)^{-R/C_v} + \frac{3\varphi - 1}{3\varphi}\right]$$

which contains no arbitrary constant except S_c, but does contain the factor a/b which depends upon the properties of the substance. At the critical state

$$U_c = \frac{a}{b}\left[\frac{8C_v}{27R} + \frac{2}{3}\right]$$

which enables us to express U entirely in terms of the properties of the critical state.

$$\frac{U}{U_c} = \frac{\frac{8}{27}\frac{C_v}{R} e^{\frac{S-S_c}{C_v}} \left(\frac{3\varphi - 1}{2}\right)^{-\frac{R}{C_v}} + \frac{3\varphi - 1}{3\varphi}}{\frac{8}{27}\frac{C_v}{R} + \frac{2}{3}}$$

or letting

$$\varphi = x$$

$$(S - S_c)/C_v = y$$

$$U/U_c = z$$

$$z = \frac{\frac{8}{27}\frac{C_v}{R}e^{y}\left(\frac{3x-1}{2}\right)^{-\frac{R}{C_v}} + \frac{3x-1}{3x}}{\frac{8}{27}\frac{C_v}{R} + \frac{2}{3}}.$$

The general properties of this equation are very much the same for the different values of the ratio $\frac{R}{C_v}$. For my computations I have selected the value 2/7, suggested by the Kinetic Theory for such triatomic gases as H_2O and CO_2. For these it becomes.

$$z = \frac{14}{23}e^{y}\left(\frac{3x-1}{2}\right)^{-\frac{7}{2}} + \frac{9}{46}\frac{3x-1}{x}. \tag{9}$$

The general form of the surface is shown by Fig. 2, sections by planes of constant entropy, $y = \text{const.}$

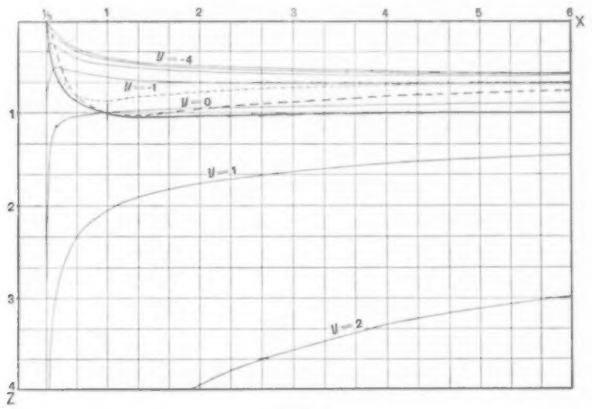

Fig. 2.

ISOTHERMALS.

Equation (7) becomes, dividing both members by C_v and substituting the new variables,

$$y = \log\vartheta + \frac{R}{C_v}\log\frac{3x-1}{2} \tag{10}$$

$$= \log\vartheta + \frac{2}{7}\log\frac{3x-1}{2}.$$

These curves are shown in Fig. 3 for the same values of ϑ which were employed in Fig. 1.

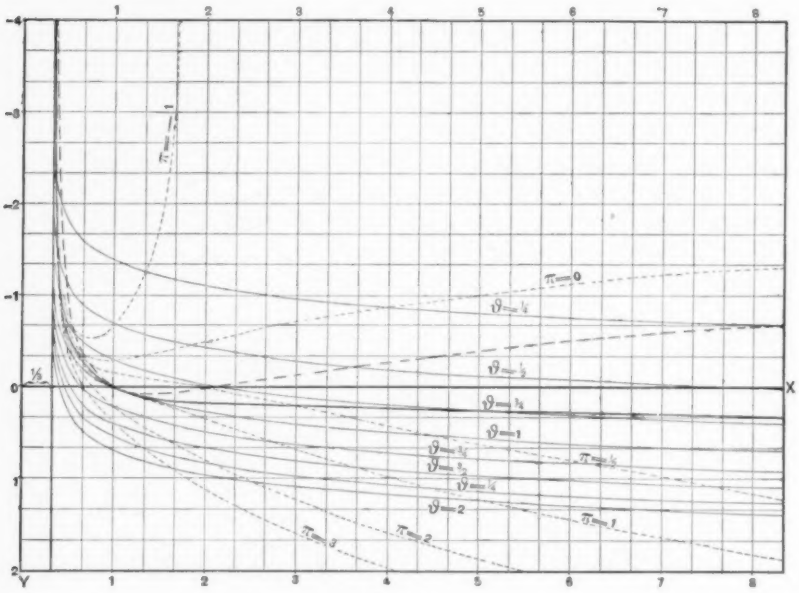

Fig. 3.

ISOPIESTICS.

Similarly substituting in (2) the value of ϑ in (8), and changing the coördinates, we obtain as the equation of the isopiestics,

$$(11) \qquad y = \log\left(\pi + \frac{3}{x^2}\right) + \frac{9}{7}\log\frac{3x-1}{2} - \log 4.$$

These curves also are shown in Fig. 3.

LINE OF INSTABILITY.

The unstable and explosive condition corresponding to certain states of a substance following van der Waals' equation is characterized by the fact that when the temperature is kept constant an increase of volume is accompanied by an increase of pressure, and vice versa, or analytically

$$\frac{\partial p}{\partial v}_{(T=\text{Const})} > 0.$$

The curve limiting this unstable region will then have the differential equation

$$\frac{\partial \pi}{\partial \varphi}_{(\vartheta)} = 0.$$

It is most convenient to obtain this directly from (2), writing

$$\pi = \frac{8\vartheta}{3\varphi - 1} - \frac{3}{\varphi^2},$$

$$\frac{\partial \pi}{\partial \varphi} = -\frac{3 \cdot 8\vartheta}{(3\varphi - 1)^2} + \frac{6}{\varphi^3} = 0,$$

$$4\vartheta = \frac{(3\varphi - 1)^2}{\varphi^3}.$$

Eliminating ϑ between this equation and (2) we have

$$\pi = \frac{2(3\varphi - 1)}{\varphi^3} - \frac{3}{\varphi^2}, \tag{12}$$

the equation of the broken line in Fig. 1. Eliminating instead with (8), and making the same substitutions as before, we have

$$e^y = \frac{1}{x^3}\left(\frac{3x - 1}{2}\right)^{\frac{16}{7}}$$

$$y = \frac{16}{7} \log \frac{3x - 1}{2} - 3 \log x, \tag{13}$$

or eliminating y by the equation of the surface, (9)

$$z = \frac{14}{23}\frac{1}{x^3}\left(\frac{3x - 1}{2}\right)^2 + \frac{9}{46}\frac{3x - 1}{4}, \tag{14}$$

whose projections are given by the broken lines in Figs. 2 and 3, respectively.

Region of Two Phases.

It is a familiar fact that while van der Waals' Equation, apparently applying to both the liquid and vapor phases, suggests only such transformations from one phase to the other as shall keep

the substance considered in one homogeneous phase, the actual transformations are usually effected by the discontinuous process of vaporization, during which the substance is non-homogeneous, being present in both the liquid and vapor phases at the same time. The actual isothermal of this process is represented in figure 1 for the temperature $\vartheta = \frac{7}{8}$ by a horizontal straight line, intersecting the ideal isothermal in three points. In Gibbs' Model, as Maxwell has shown,[1] such an isothermal is given by a line connecting the two points of tangency of a plane resting upon the model. We may then attempt to locate the boundary line separating the regions representing a single phase from those referring to the co-existence of two phases by either of two methods.

If the area under the actual and under the ideal isothermals is to be the same, if we call the volume of the liquid just ready to vaporize φ_1, and that of the saturated vapor φ_2, both at the temperature ϑ and the pressure Π, then

$$\int_{\phi_1}^{\phi_2} \Pi d\varphi = \int_{\phi_1}^{\phi_2} \pi d\varphi = \int_{\phi_1}^{\phi_2} \left(\frac{8\vartheta}{3\varphi - 1} - \frac{3}{\varphi^2} \right) d\varphi$$

or integrating,

$$\Pi(\varphi_2 - \varphi_1) = \frac{8}{3}\,\vartheta \log \frac{3\varphi_2 - 1}{3\varphi_1 - 1} + 3\left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1} \right)$$

and since van der Waals' equation itself still applies to the saturated vapor and liquid

$$\Pi = \frac{8\vartheta}{3\varphi_2 - 1} - \frac{3}{\varphi_2^2}$$

$$\Pi = \frac{8\vartheta}{3\varphi_1 - 1} - \frac{3}{\varphi_1^2}$$

Three equations which are sufficient to give the relations between Π, φ_1, φ_2 and ϑ if we can solve them. If we do obtain the solution, we can plot the locus of φ_1 and φ_2 in Fig. 1, and by algebraic transformations find the corresponding locus in Fig. 2, and Fig. 3.

Or, we may attempt the direct solution. The equation of the surface of our model (9), is of the form

[1] Theory of Heat, pp. 195–205.

$$z = me^{y}\left(\frac{3x-1}{2}\right)^{-r} + n\frac{3x-1}{x}$$

The equation of a plane is

$$z = ax + by + c.$$

If now the plane touch the surface at two points $x', y', z', x'', y'', z''$, the coördinates of these points must satisfy both equations, giving us

$$z' = me^{y'}\left(\frac{3x'-1}{2}\right)^{-r} + n\frac{3x'-1}{x'},$$

$$z'' = me^{y''}\left(\frac{3x''-1}{2}\right)^{-r} + n\frac{3x''-1}{x''},$$

$$z' = ax' + by' + c,$$
$$z'' = ax'' + by'' + c.$$

The fact that the plane and surface are mutually tangent makes the slopes in the principal directions the same, or

$$\frac{\partial z'}{\partial x'} = a = -\frac{3}{2}mre^{y'}\left(\frac{3x'-1}{2}\right)^{-1-r} + n\frac{3x'-1}{x'},$$

$$\frac{\partial z''}{\partial x''} = a = -\frac{3}{2}mre^{y''}\left(\frac{3x''-1}{2}\right)^{-1-r} + n\frac{3x''-1}{x''},$$

$$\frac{\partial z'}{\partial y'} = b = me^{y'}\left(\frac{3x'-1}{2}\right)^{-r},$$

$$\frac{\partial z''}{\partial y''} = b = me^{y''}\left(\frac{3x''-1}{2}\right)^{-r}.$$

A point xyz which lies on the line connecting the two points of tangency will satisfy the two equations

$$\frac{y-y'}{x-x'} = \frac{y'-y''}{x'-x''},$$

$$\frac{z-z'}{x-x'} = \frac{z'-z''}{x'-x''},$$

which gives us ten equations between which we may eliminate the six coördinates, $x', y', z', x'', y'', z''$ and the three constants a, b, c, still leaving an equation involving x, y, z, which is the equation of

the developable surface generated by a plane rolling on the model. The intersection of this surface with our model will be the line desired, separating the regions relating to one phase from that relating to two phases.

These two analytical methods of solution are here given for the sake of completeness. A direct and complete solution by algebraic methods does not seem possible, and lack of time has prevented my attempting approximate or graphical methods of solution. The horizontal portion of the isothermal $\vartheta = 3/4$, in Fig. 1, was actually drawn by the method of trial, so as to make the areas included between it and the two segments of the ideal isothermal equal. This

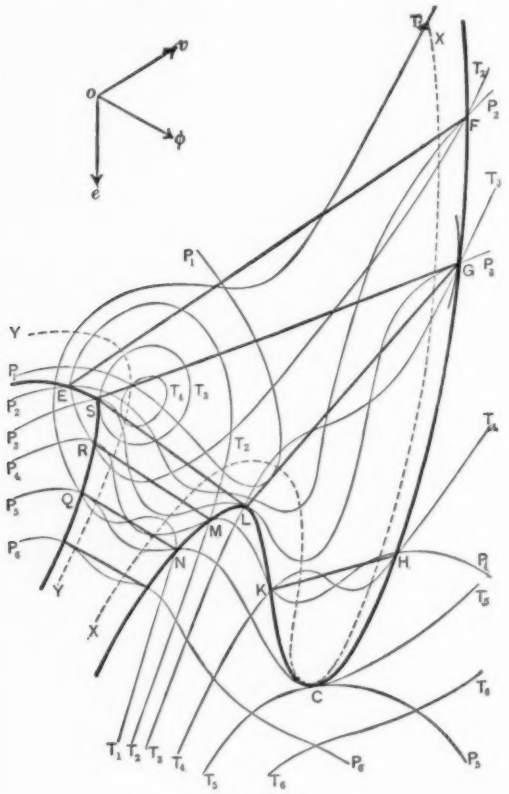

Fig. 4.

enabled the line of separation to be drawn of approximately the right position and form.

Discussion.

The lines of the various figures form a network which is to be considered as spread out over the surface of either model. These lines are lines of constant volume, constant pressure, constant temperature, and constant entropy. Fig. 1 is slightly less complex than Fig. 3, lacking the system of isentropics. Except for this there must be a one to one correspondence between the two net-works, since they represent identically the same properties, thus the same lines pass through the critical point in each, and in each the same lines have points of tangency or intersection for the same values of x.

In the more specific discussion of the lines of Fig. 3, it is well to compare it with Fig. 4, which is taken from Maxwell.[1] An important difference which must be taken into account is that Fig. 3, refers only to the liquid and gaseous states, while Fig. 4 has also a region referring to the solid state.

The *isothermals* are indicated in Fig. 3 by the letter ϑ, in Fig. 4 by the letter T. An adiabatic or isentropic increase of volume is, so far as I know, under all circumstances accompanied by a fall of temperature. This property appears in Fig. 3 in which the isothermals slope downward to the right, but not in Fig. 4, in which they slope in general upward to the right, but occasionally have singular forms.

The *isopiestics* are indicated in Fig. 3 by the letter π, in Fig. 4 by the letter P. The general form of a line of constant negative pressure it given by the curve $\pi = -1$. It has either two or no intersections with an isothermal, being tangent in one special case for each isopiestic. This property appears from Fig. 1. Each negative isopiestic passes off to infinity becoming asymptotic to a vertical line. The one shown has for its asymptote the line $x = \sqrt{3}$. The limiting case is the line $\pi = 0$. Like all the other isopiestics it is asymptotic to the line $x = \frac{1}{3}$. It cuts all the isothermals twice or not at all, being tangent to one particular isothermal, and approaches parallelism to the isothermals, after cutting above them all, for $x = \infty$. This corresponds to the fact that in Fig. 1 the axis $\pi = 0$ is an asymptote to all the isothermals. Another limit-

[1] Theory of Heat, p. 207.

ing case is the line $\pi = 1$, which is tangent to the isothermal $\vartheta = 1$ at the critical point, and cuts every other isothermal once only. An example of an intermediate case is the isopiestic $\pi = \frac{1}{2}$, which cuts every isothermal either once or three times. The other case is illustrated by the lines $\pi = 2$ and $\pi = 3$ which cut every isothermal once only. All these properties appear from figure 1, and the fact that they also appear in Fig. 3, may be taken as a check upon the accuracy of the drawing. In general, the isopiestics for positive pressures slope downward to the right, although for very small pressures less than $\pi = \frac{1}{2}$ they may in the middle of the field slope up to the right. This does not appear in Fig. 4, which also does not exhibit the bunching of lines asymptotic to $x = \frac{1}{3}$. The line P_1, which represents a negative pressure, is not entirely unlike the line $\pi = -1$.

The *Line of Instability*, represented by the broken line in Fig. 3, and by the dotted line in Fig. 4, has in the two cases not dissimilar properties, but in the former it dips below the level of the critical point, and passes off to infinity *horizontal.*

The line dividing the regions corresponding to a single phase from that corresponding to two co-existant phases is represented in both figures by a heavy line, which passes through the critical point and is there tangent to the critical isothermal, critical isopiestic, and the line of instability. It cuts all the isothermals below the critical twice, and all the positive isopiestics below the critical twice. In figure 3 it lies entirely between the isopiestics $\pi = 0$ and $\pi = 1$. At the critical point it passes below the level of the critical point, and passes off to infinity horizontal, as do the isothermals. It appears immediately from this diagram that the adiabatic expansion of a saturated vapor would result in partial condensation. Whether this line would in other regions at lower temperatures have such a form that adiabatic expansion would superheat a saturated vapor could only be told if we were able to obtain an analytic solution for the line, or plot it point by point. The matter is not of great importance, as we are dealing only with an equation, and not an actual substance.

The lines of Fig. 3 have all been plotted from their equations, with the exception of the one last discussed, which, however, we

have been able to draw with fair accuracy from our knowledge of its relations to the other lines of the figure. If the axes of co-ordinates of Fig. 4 are to be considered as similarly situated, the discrepancy between the two is very considerable, and very troublesome. If, however, the axes are to be taken as obliquely as they are drawn, v corresponding to x and φ to y, or if we regard Fig. 4 as a perspective drawing rather than a projection on one of the principal planes, the discrepancies, while by no means disappearing, become much less.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIFORNIA, July 2, 1900.

ON THE FORMS OF CURVES PRESENTED BY THE MICHELSON INTERFEROMETER.

BY JOHN CUTLER SHEDD.

PRELIMINARY.

THE theory of the interference phenomena produced by this instrument has been entered into somewhat in detail by Professor A. A. Michelson.[1] In the present paper it is proposed, first to follow the ground plan presented by Professor Michelson and then to give a point of view which is believed to be novel.

PART I.

In Fig. 1 is given a plan view of the usual form of interferometer.

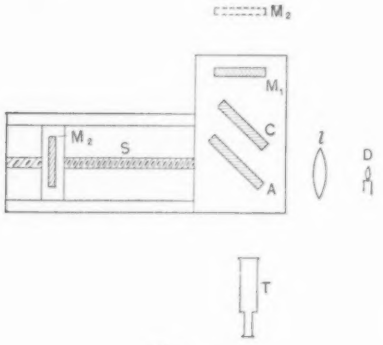

Fig. 1.

$M_1 M_2 \equiv$ heavily silvered mirrors.

$A \equiv$ plate, half silvered on side toward M_1.

$C \equiv$ plate of same thickness as A and placed parallel to it. Plate C is called the compensator.

M_2 is on a movable carriage controlled by screw S.

The interference pattern is viewed by the eye, or telescope, placed at T.

[1] Phil. Mag. [5], 13, 1882, pp. 236–242.

It is evident that the mirror M_2 may be regarded as occupying a position behind M_1, as shown in the dotted lines. Regarding M_2 as so placed, we may, as has been shown by Professor Michelson, suppress everything excepting M_1 and M_2.

Let de be drawn parallel to the line of intersection of the planes of M_1 and M_2 and let the angle of intersection be φ.

The whole phenomenon may now be discussed on the basis of a source of light S placed in the plane of M_1 and of its image in M_2.

In this discussion the following notation will be used:

$t_0 \equiv$ Perpendicular distance between M_1 and M_2. On the instrument this is controlled by the screw S (Fig. 1).

$\varphi \equiv$ Angle between M_1 and M_2. This is controlled by adjusting screws at the back of M_1.

$P \equiv$ Perpendicular distance from M_1 to the plane containing the interference pattern. This plane of maximum distinctness is called the focal plane.

$\theta \equiv$ Angle subtended by the source of light at point P on the focal plane, in a plane perpendicular to the plane of φ.

$i \equiv$ Same in plane parallel to plane of φ.

$\Delta \equiv$ Difference of path traversed by the two rays producing interference at P.

Professor Michelson has shown that the value of Δ is given by the equation

$$\Delta = 2\frac{t_0 + P \tan i \tan \varphi}{\sqrt{1 + \tan^2 i + \tan^2 \varphi}}. \qquad (1)$$

In general Δ has all values but the value which will give the most distinct fringes is given by the conditions

$$\frac{\partial \Delta}{\partial \theta} = 0$$

and

$$\frac{\partial \Delta}{\partial i} = 0.$$

This gives the value of P for which the phenomenon of interference is most distinct. The value is

$$P = \frac{t_0}{\tan \varphi} \tan i. \qquad (2)$$

It follows from equation (2) that if $t_0 = 0$ or $i = 0$ then $P = 0$ and the interference fringes are on the surface of M_1. If $t_0 = 0$ and $\varphi = 0$, or $i = 0$ and $\varphi = 0$ then P is indeterminate. If $\varphi = 0$ then t_0 is everywhere uniform and $P = \infty$. If φ and i have same sign P is positive and the fringes are in front of M_1. If φ and i have opposite signs P is negative and the fringes are behind M_1.

If M_2 is behind M_1, then φ is regarded as positive and t_0 as positive. If M_2 is in front of M_1 φ and t_0 are both regarded as negative. The plane containing the interference pattern is called the *Focal* plane, and upon this plane the eye or telescope must be focused in order to get the most distinct view of the fringes. The Focal plane is parallel to the plane of M_1 and at a distance P from it.

Any point on the Focal plane is determined by the equations

$$\left.\begin{aligned} x &= P \tan i \\ y &= P \tan \theta \end{aligned}\right\} \qquad (3)$$

and the form of curve produced in the interference pattern is found by substituting equations (3) in equation (1). This gives the following equation of the interference curves:

$$\Delta^2 y^2 = (4P^2 \tan^2 \varphi - \Delta^2)x^2 + 8t_0 P^2 \tan \varphi x + P^2(4t_0^2 - \Delta^2). \qquad (4)$$

The further discussion of the subject is an analysis of equation (4), and it is very essential to its good understanding that every term involved be thoroughly apprehended. We therefore repeat:

$2t_0 \equiv$ Perpendicular distance between M_1 and its image in M_2, optically considered [this reduces to the value of Δ when i and θ both are zero].

$\varphi \equiv$ angle that M_1M_2 make with each other [physically the angle by which they differ from 90°].

$P \equiv$ Perpendicular distance of the focal plane from M_1.

$i, \theta \equiv$ Angles of incidence upon the focal plane, of the interfering beams: i, being measured in the plane parallel to and θ in the plane normal to the plane of φ. The focal plane is normal both to the plane of i and of θ.

$\Delta =$ Difference of path of the interfering beams.

In the article already referred to Professor Michelson expresses equation (4) in the equivalent form

$$\Delta^2 y^2 = (4S^2K^2 - \Delta^2)x^2 + 8TSKDx + (4T^2 - \Delta^2)D^2 \qquad (5)$$

and then points out what curves are represented by equation (5) and the values of the constants necessary in each case.

It would seem to be equally as simple and more satisfactory to refer directly to equation (4). The following are the principal cases that arise.

(1) $\Delta = 0$; Equation (4) becomes

$$x = \frac{t_0}{\tan \varphi} \qquad (6)$$

representing a straight line.

(2) $\varphi = 0$; then

$$x^2 + y^2 = \frac{P^2}{\Delta^2}(4t_0^2 - \Delta^2) \qquad (7)$$

giving circular fringes.

(3) $\Delta = 2P \tan \varphi$. In this case equation (4) becomes

$$y^2 = \frac{4Ft_0}{\Delta} x_1. \qquad (8)$$

In equation (8) this axis of y has been shifted so that

$$x_1 = x - P\frac{4t_0^2 - \Delta^2}{4t_0\Delta}$$

thus giving the central equation. Equation (8) is that of a parabola

(4) $\Delta > 2P \tan \varphi$; Equation (4) takes the form

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \qquad (9)$$

where

$$A^2 = \frac{\Delta^2 P^2(4t_0^2 - \Delta^2)}{\Delta^2 - 4P^2 \tan^2 \varphi}$$

and

$$B^2 = P^2(4t_0^2 - \Delta^2)$$

thus giving the semi-axes of the ellipse

(5) $\Delta < 2P \tan \varphi$. In this case the hyperbola

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \qquad (10)$$

is obtained where

$$A^2 = \frac{\Delta^2 P^2(\Delta^2 - 4t_0^2)}{\Delta^2 - 4P^2 \tan^2 \varphi}$$

and

$$B^2 = P^2(\Delta^2 - 4t_0^2).$$

A more detailed consideration of these cases is reserved for the second part of the paper.

PART II.

The curves just considered may also be very readily developed by considering their eccentricity. The general expression for the eccentricity of equation (4) is

$$e = \sqrt{1 + A/B}$$

where A is the coefficient of x^2 and B of y^2. In the present case this becomes the very simple expression

$$e = \frac{2P \tan \varphi}{\Delta}. \tag{11}$$

Also since in equation (2)

$$P = \frac{t_0}{\tan \varphi} \tan i$$

we get

$$e = \frac{2t_0 \tan i}{\Delta}. \tag{12}$$

Finally substituting the value of Δ given in equation (1) we get

$$e = \sin i \cos i \sqrt{1 + \tan^2 i + \tan^2 \theta} \tag{13}$$

or

$$e = \sin i \sqrt{1 + \tan^2 \theta \cos^2 i}. \tag{14}$$

The various cases that rise can be readily covered by equations (11) to (14). The following will be considered.

Case 1. $\Delta = 0$; then by equation (11) or (12)

$$e = \infty \tag{15}$$

and the curves are straight lines. If, however, t_0 be also zero, then by equations (11) or (12) e is indeterminate. The meaning of this is that mirrors M_1 and M_2 coincide and hence there is but a single

source of light. The conditions therefore are not favorable for interference.

However, if $t_0 = 0$ and $\varphi > 0$ then $P = 0$ and it is also apparent that both i and θ are zero. If i and θ are both zero it is apparent by equation (13) that $e = 0$, and the curves if existent, would be circles. This point will be returned to later.

Case 2. $t_0 = 0$ and $\varphi > 0$; then $P = 0$, and the central band will be a straight line (corresponding to $\Delta = 0$ and $e = \infty$) flanked on either side by bands of opposite curvature ($\Delta > 0\ e < \infty$). The fringes on one side are behind the mirror (P negative) while those on the other side are in front of the mirror (P positive).

Referring to equations (1) and (11) viz. :

$$\Delta = 2\frac{t_0 + P\tan i\tan\varphi}{\sqrt{1+\tan^2 i+\tan\theta}} \qquad (1)$$

and

$$e = \frac{2P\tan\varphi}{\Delta} \qquad (11)$$

we see that since P is small, being zero for the central band, the values of i and θ rapidly increase, hence (t_0 also being very small) the value of Δ is small (Eq. 1) and the value of e (Eq. 11) will be very large. The larger the value of φ the more rapidly does i change (i being in the plane of φ) and the more slowly does the value of e fall. Furthermore the central straight band lies parallel to the intersection of M_1 and M_2 and hence the more rapidly i changes (*i. e.*, the larger the value of φ) the closer together will the fringes be. Hence as φ is made to increase the field becomes covered with narrow fringes that are practically straight lines. If now M_2 be made to recede, so as to increase t_0, the fringes soon appear curved. Again increasing the value of φ tends to make the fringes straight, but a limit is soon reached where they are too narrow to be distinguishable.

For work over a limited range of t_0, it is frequently convenient to use straight fringes. It is also apparent that under these conditions P is practically zero and hence the telescope is to be focused on mirror M.

This point is very well illustrated in the work of Morley and Rogers in their research on the thermal expansion of metals.[1]

Case 3. $\varphi = 0$. In this case $e = 0$ and the fringes are circles. If $\varphi = 0$ then $P = \infty$ (Eq. 2) and the telescope must be focused for parallel light. Regarding equation (7) we see that the radius of each circle is

$$R = \frac{P}{\Delta}\sqrt{(2t_0 + \Delta)(2t_0 - \Delta)}.$$

But $2t_0 - \Delta = n\lambda$ and $2t_0 + \Delta = 2\Delta$ (approximately), so that

$$R = P\sqrt{\frac{n\lambda}{t_0}} \quad \text{(approximately)}.$$

$\therefore R \propto \sqrt{n}$ showing that the circular fringes follow the same law as do Newton's rings.

Case 4. $\varphi > 0$, but small. We obtain by equation (6), if we put $2t_0 = \Delta$, and from equation (12)

$$e = \tan i. \tag{15}$$

For small values of i the fringes are circular, becoming elliptical as i increases. Thus for values of $i < 5°$, $e < 0.1$.

It is easy to so adjust t_0 and φ that circular fringes shall be seen in the center of the field of view, surrounded by increasingly elliptical fringes as i increases in value.

In this connection a very interesting case arises. For large values of t_0, φ and i being of opposite sign, the focal plain is behind M_1. Under these conditions t_0 and φ are positive, i and P negative. By equation (2) Δ is negative and e, of course, positive. Also for large values of t_0 the value of P given by equation (3) is large, and the value of i is necessarily very small, hence e is small (Eq. (12)), and the fringes are circles. If now M_2 be advanced until $t_0 = 0$ then $P = 0$ (Eq. (2)) and the focal plane is on M_1, or, more correctly, the focal plane intersects M_1 in the field of vision. At this point $e = \tan i$ (practically) and the fringes are still circles. Since $t_0 = 0$ at this point circular chromatic fringes may be obtained. These fringes are beautifully distinct and brilliant.

[1] PHYSICAL REVIEW, Vol. IV, p. 7, 1896.

If M_1 be still further advanced so as to bring M_2 in front of M_1, then t_0 and φ change sign, becoming negative; i and P become positive, while Δ remains negative.[1]

The value of Δ then becomes

$$\Delta = 2\frac{-t_0 + P\tan i\tan\varphi}{\sqrt{1+\tan^2 i+\tan^2\varphi}}$$

and $t_0 > P\tan i\tan\varphi$, so that Δ is negative. As M_1 is still further advanced t_0 increases in value and so also P. For a time the value of Δ will diminish and the value of e (Eqs. 11 or 12) increase. The result is that the curves rapidly change from circles ($e=0$) t_0 ellipses ($e>0$ but <1) then to the parabola ($e=1$) becoming hyperbolæ ($e>1$) and finally giving the straight line[2] ($e=\infty$).

If after the minimum value of Δ is reached t_0 be still further increased in value, the increased value of P is more than counterbalanced by the diminished value of tan i, so that the term $P\tan i\tan\varphi$ diminishes in value. The value of Δ therefore increases and the value of e diminishes. The result is that when M_2 is further advanced the curves change curvature (the conjugate hyperbolæ coming into view) and pass through the hyperbolic, parabolic, elliptic forms and finally again appear as circles.

The foregoing case furnishes a very ready method of finding the chromatic fringes (zero position of the instrument) without any previous acquaintance with the instrument. The scheme is as follows: Having secured fringes with the sodium flame in the usual manner, run M_2 out 6 or 7 mm. and adjust M_2 so that the parallax is a minimum. This will give a small value of φ. Now advance M_2 through the zero position until the curves are hyperbolæ. Finally reverse the motion and let M_2 recede until the fringes just begin to become circular. If, now, white light be substituted for the sodium flame, and M_2 be made to slowly recede by means of the

[1] The negative sign before Δ indicates that the increment of Δ, as i increases, is a diminishing increment. The result is that the focal plane becomes a curved surface concave to the observer for negative values of P (P diminishing as i increases in value). This can be detected by the eye, the fringes appearing to lie on a concave surface.

[2] This will depend upon whether Δ becomes zero or merely reaches a minimum value and then increases again. On the instrument Δ may be controlled, and the fringes made to be practically straight lines.

tangent screw, chromatic fringes quickly appear. So distinct are they that the diffused light of the room is sufficient to give them very clearly.

Case 5. $\Delta = 2\,P \tan \varphi$ or $\Delta = 2\,t_0 \tan \varphi$.

In this case $e = 1$. (16)

Putting this value in equation (13) we obtain

$$1 = \sin i \cos i \sqrt{1 + \tan^2 i + \tan^2},$$

which may be reduced to the form

$$\sin \theta = \frac{1}{\sqrt{1 + \sin^2 i}}. \tag{17}$$

The limiting values of θ and i are

$$i = \pm 90°, \text{ and } \theta = 90° \pm 45°.$$

It is thus evident that the parabola cannot be seen unless $\theta > 45°$; and it is equally evident that θ cannot have this value unless the focal plane is quite close to M_1. This explains why the parabola is only seen for small values of t_0; or large values of φ, giving small values for P.

Case 6. $\Delta < 2P \tan \varphi$ or $\Delta < 2t_0 \tan i$.

Then $e > 1$, and the curves are hyperbolæ. What has been said under Case 5, as to the value of t_0 and φ holds equally for this case, showing that the hyperbolic curves are only observed when the value of t_0 is very small.

Case 7. The Equilateral Hyperbola.

In this case $e = \sqrt{2}$ and equation (14) reduces to

$$\tan \theta = \sqrt{\frac{2}{\sin^2 i} + \sin^2 i - 1}. \tag{18}$$

The following table based upon equation (18) shows the relation of i and θ.

If i and θ be interchanged the conjugate system of hyperbolæ is obtained. In the general case of the hyperbola the minimum value of θ was found to be 45°. In the equilateral hyperbola the minimum value is 54°, showing that in this case the range of t_0 is still more restricted than in the general case.

TABLE I.

$\sin i$	i	$\tan \theta$	θ	P
0	0°	∞	90°	$\frac{t_0}{\tan \phi} \times 0 = 0$
0.5	30°	2.7	70°—	$\frac{t_0}{\tan \phi} \times 0.58$
0.707	45°	1.87	62°—	$\frac{t_0}{\tan \phi} \times 1$
0.877	60°	1.58	58°—	$\frac{t_0}{\tan \phi} \times 1.73$
1	90°	1.4	54°40′	$\frac{t_0}{\tan \phi} \infty = \infty$

Case 7. Rotation of the Compensator. It is very instructive, for a given setting of the mirror M_2, to slowly rotate the compensating plate (C, Fig. 1). The effect of such rotation is to change the length of path through the glass, and hence to alter the value of t_0. If the value of t_0 be already large, this change will not appreciably effect either the value of P (Eq. 2) or that of Δ (Eq. 1), and hence the value of e remains sensibly unaltered.

If, however, t_0 be small the rotation of the compensator will produce a large relative change in t_0 and hence a large change in both Δ and P, resulting in a rapid change in the value of e.

For example, if the value of t_0 be small, and the compensating plate be parallel to plate A; then on rotation such as to diminish the optical path through the glass, the circular fringes are seen to change to the elliptical form, the value of e becoming greater than zero.

As t_0 becomes still smaller, due to further rotation of the compensator, e increases in value, and, corresponding to this the fringes pass through the Parabolic, into the hyperbolic form. Soon the conjugate system of hyperbolæ appear in the field of view, presenting a very pretty sight. If the compensator be rotated until it becomes parallel to M_1, the behavior of the fringes changes. As the parallel position is passed the direction of motion of the fringes reverses, and the value of t_0 begins to increase. The reversal in the movement of the fringes furnishes a very sensitive test for the parallelism of plates C and M_1 (Fig. 1). This test for parallelism

might, from its sensitiveness, be made of more than passing value in other lines of work.

Case 8. $\varphi = 0$; $t_0 = 0$. Under these conditions P is indeterminate (Eq. 2). Therefore Δ is indeterminate (Eq. 1), and so also e by equations (11) or (12). If $t_0 = 0$ it is apparent that M_1 and M_2 coincide, and, as has already been pointed out, there should be no interference and hence no interference pattern.

This however will depend upon whether the two beams of light are exactly in phase. Now since one beam suffers internal and the other external reflection at plate A, if a difference of phase other than $\frac{1}{2}\lambda$ takes place interference would result and an interference pattern would be seen. Since M_1 and M_2 coincide the values of i and θ would seem to be 0°, and the value of e by equation (13) would then be

$$e = \sin i \cos i \sqrt{1 + \tan^2 i + \tan \theta}$$

$$e = 0 \qquad (19)$$

and hence the fringes would be circles. Or we may consider that a difference of phase between the two beams, due to any cause, would be equivalent to a finite value for t_0. Then since $\varphi = 0$, e would be zero by equation (11).

In the case of white light, in addition to the difference of phase due to reflection, the compensating plate C introduces a *dispersion*[1] effect (excepting when C is strictly parallel to A) thus (in general) adding to the phase difference.

The result is that very brilliant chromatic *circular* fringes are to be seen even when $\varphi = 0$ and $t_0 = 0$. The presence of fringes furnishes a test for the presence of a phase difference produced by reflection. Under these conditions as many as 147 circular chromatic fringes have been counted.

Equation (13), perhaps, sums up all of the preceding cases in as concise a manner as is possible. When t_0 is large it is evident that i and θ must both be very small and hence $e = 0$ (practically). As t_0 becomes smaller the possible values of i and θ are larger giving the successively larger values for e.

[1] This is the "chromatism" investigated by Conn, C. R. t, xciii, 1881 and by Michelson, Phil. May [5] 13, 236, 1882.

It is fascinating to observe the transformation of the interference fringes as they pass from one form of curve to another. Not only are the phenomena very beautiful in themselves but there is no better way of becoming thoroughly conversant with this admirable and many-sided instrument than is furnished by the study of the manifold curves presented by the interferometer. For the verification of the deductions given in this paper, I wish to thank Mr. A. C. Scott, of the staff of the Agricultural College, Kingston, R. I., who during the past summer collaborated with me in the physical laboratory of the University of Wisconsin.

PHYSICAL LABORATORY, COLORADO COLLEGE.

NEW BOOKS.

The Strength of Materials. By J. A. EWING. Cambridge University Press; New York, The Macmillan Co., 1899. Pp. xii+245.

The work above referred to was printed at the University press, Cambridge, in 1899, and as we learn from the preface, is intended principally to set forth briefly a lecture room treatment of the subject which is to be supplemented by laboratory and drawing office work.

The book contains twelve chapters and an appendix and constitutes a most excellent treatise on the subject. So far as it relates to the application of mathematics to the strength of materials it is decidedly the best work published, for the student. Throughout the entire book the distinction between stress and strain is made prominent, and although the terms are used correctly they are used in an entirely different sense from that of some American writers on the subject. Stress is defined as the mutual action between two bodies or between two parts of a body whereby each of the two exerts a force upon the other. Strain is defined as a change of shape produced by stress. Many of the American writers use the word *strain* in the sense of stress as defined above. The distinction made by Professor Ewing is to be commended, as it is believed that the theory relating to the strength of materials would be much simplified if all writers would adopt these terms and use them in this manner.

The theoretical discussions are stated with admirable clearness and the book covers the broad mathematical field relating to strength of materials and its application to frames and simple trusses thoroughly. The method of treatment makes it well adapted for use as a text-book.

A chapter is devoted to the Testing of Materials in which considerable information is given regarding testing machines. This chapter is the only one which is open to criticism from an American standpoint. All the testing machines illustrated are either of an antiquated type or are those manufactured by English builders. The forms are, from an American standpoint, unmechanical, unwieldy and awkward to use although doubtless giving accurate results. It would seem that a text-book intended for general use should present information pertaining to other forms of testing machines, especially when it is considered that the forms shown have only a very limited use and are confined to a country which at the present time is manufacturing only a very small fraction of the steel used for structural purposes by the world, and moreover this fraction is constantly diminishing.

On the whole the book is a valuable one as it presents in a small compass all the important principles and the results of the principal investigations made on this important subject.

R. C. CARPENTER.

INDEX TO VOLUME XI.

A.

Acetylene flame, efficiency of, *Nichols*, 215.

Allen, F., Effect on the persistence of vision of exposing the eye to light of various wave-lengths, 257.

American Association for the Advancement of Science, 188.

Attenuation of Sound, *Duff*, 65.

Atwood's machine and stop-clock, *Baker*, 105.

Austin, L. W., Modification of Mance's method of determining battery resistance, 117.

B.

Baker, W. C., Description of a new Atwood's machine and stop-clock, 105.

Balance, spiral spring, *Linebarger*, 110.

Becquerel rays, a résumé, *Stewart*, 155.

Bedell, F., Young's *Electrical Testing for Telegraph Engineers*, 64.

Book reviews.

- Arithmetic of chemistry, *Waddell*, 120.
- Central station electricity supply, *Gay and Yeaman*, 123.
- Electrical testing for telegraph engineers, *Young*, 64.
- Electromagnetic theory, *Heaviside*, 60.
- History of mathematics, *Fink*, 192.
- Optical activity and chemical composition, *Landolt*, 191.
- Ostwald's Klassiker, no. 109, 126.
- Outlines of industrial chemistry, *Thorp*, 121.
- Magnetic induction in iron and other metals, *Ewing*, 126.
- Physical chemistry for beginners, *van Deventer*, 120.
- Physics: Experimental and theoretical, *Jude*, 118.
- Spirit of organic chemistry, *Lachman*, 120.
- Strength of materials, *Ewing*, 316.
- Text-book on physics. Sound. *Poynting and Thomson*, 124.
- Theory of electrolytic dissociation and some of its applications, *Jones*, 127.

Boyle's law apparatus, *Rogers*, 112.

Boyle's law, verification of, *Williams*, 255.

Boynton, W. P., Gibbs' thermo-dynamic model for a substance following van der Waal's equation, 291.

Briggs, L. J., A new form of electrical condenser having a capacity capable of continuous adjustment, 14.

Buckingham, E., Jones' *Theory of Electrolytic Dissociation*, 127.

C.

Caldwell, G. C., Thorp's *Outlines of Industrial Chemistry*, 121.

Capacity capable of continuous adjustment, *Briggs*, 14.

Carhart, H. S., Thermodynamics of the voltaic cell, 1.

Carpenter, R. C., Ewing's *Strength of Materials*, 316.

Coherer, theory of, *Guthe and Trowbridge*, 22.

Condenser, electrical, having a capacity capable of continuous adjustment, *Briggs*, 14.

D.

Day, W. S., New method of measuring the resistance of a galvanometer, 251.

Dielectrics, solid, spark-length as modified by, *Humphreys*, 79.

Duff, A. W., Attenuation of sound, 65.

E.

Elasticity, effect of magnetization on the modulus of, *Stevens*, 95.

Elasticity, lecture room methods in the elementary theory of, *Franklin*, 75.

Electrostatic lines of force, method of studying, *Rogers*, 56.
Ewing, J. A., *Magnetic Induction in iron and other metals*, 126.
Ewing, J. A., *Strength of Materials*, 316.

F.

Ferric chloride, effect of temperature, colloidal ferric hydrate, and magnetic field on hydrolysis of, *Goodwin and Grover*, 193.
Films deposited by kathode discharge, electrical resistance of, *Longden*, 40, 84.
Fink, K., *History of Mathematics*, 192.
Franklin, W. S., Some lecture room methods in the elementary theory of elasticity, 75.
Stresses and strains in a long elastic hollow cylinder, subjected to internal and external pressure and to tension, 176.

G.

Galvanometer, new method of measuring the resistance of, *Day*, 251.
Gay, *Central Station Electricity Supply*, 123.
Gill, A. C., Landolt's *Optical Activity and Chemical Composition*, 191.
Goodwin, H. M., Hydrolysis of ferric chloride, 193.
Grover, F. W., The hydrolysis of ferric chloride, 193.
Guthe, K. E., Theory of the coherer, 22.

H.

Heaviside, O., *Electromagnetic Theory*, 60.
Humphreys, W. J., Spark-length as modified by solid dielectrics, 79.
Hydrolysis of ferric chloride, effect of temperature, colloidal ferric hydrate and magnetic field on, *Goodwin and Grover*, 193.

I.

Interferometer, forms of curves presented by, *Shedd*, 304.

J.

Jones, H. C., *Theory of Electrolytic Dissociation and Some of its Applications*, 127.
Jude, R. H., *Physics: Experimental and Theoretical*, 118.

K.

Kathode rays, development of, by ultra-violet light, *Merritt and Stewart*, 230.
Knipp, C. T., Surface tension of water above 100° C., 129.

L.

Lachman, A., *The Spirit of Organic Chemistry*, 120.
Landolt, H., *Optical Activity and Chemical Composition*, 191.
Linebarger, C. E., A new form of the spiral spring balance, 110.
Longden, A. C., Electrical resistance of thin films deposited by cathode discharge, 40, 84.

M.

Magnetization, effect of, on the modulus of elasticity, *Stevens*, 95.
Mance's method for determining battery resistance, a modification, *Austin*, 117.
Merritt, E., The development of kathode rays by ultra-violet light, 230.
Thomas Preston, 188.
Miller, G. A., Fink's *History of Mathematics*, 192.
Mixtures, spectra of, *Rollefson*, 101.

N.

Nichols, E. L., Ewing's *Magnetic Induction in Iron and Other Meta s*, 126.
Jude's *Physics: Experimental and Theoretical*, 118.
Ostwald's *Klassiker der Exacten Wissenshaften, No. 109*, 126.
Poynting and Thomson's *Text-book on Physics, Sound*, 124.
Preliminary note on the efficiency of the acetylene flame, 215.
Norris, H. H., Guy and Yeaman's *Central Station Electricity Supply*, 123.

P.

Persistence of vision, effect of various wave-lengths on, *Allen*, 257.
Photometry of glow lamps, universal apparatus for, *Sharp*, 181.
Poynting, *Text-book on Physics, Sound*, 124.
Preston, Thomas, 188.

R.

Resistance, electrical, of thin films deposited by cathode discharge, *Longden*, 40, 84.
Resistance of galvanometer, new method of measuring, *Day*, 251.
Reticles, preparation of, *Todd*, 59.
Rogers, F. J., Apparatus for demonstrating Boyle's law, 112.
Method of studying electrostatic lines of force, 56.
The M. K. S. system of units, 115.
Rollefson, C. J., Spectra of mixtures, 101.

S.

Searle, G. F. C., Heaviside's *Electromagnetic Theory*, 60.
Sharp, C. H., Universal apparatus for the photometry of glow lamps, 181.
Shedd, J. C., Forms of curves presented by the Michelson interferometer, 304.
Sound, attenuation of, *Duff*, 65.
Spark-length as modified by solid dielectrics, *Humphreys*, 79.
Spectra of mixtures, *Rollefson*, 101.
Stevens, J. S., Effect of magnetization on the modulus of elasticity, 95.
Stewart, O. M., Becquerel rays, a résumé, 155.
Development of kathode rays by ultra-violet light, 230.
Stop-clock and Atwood's machine, *Baker*, 105.
Stresses and strains in a long, elastic, hollow cylinder, *Franklin*, 176.
Surface tension of water above 100° C., *Knipp*, 129.

T.

Thermodynamic model for a substance following van der Waal's equation, *Boynton*, 291.
Thermodynamics of the voltaic cell, *Carhart*, 1.
Thomson, J. J., *Text-book on Physics, Sound*, 124.
Thorp, F. H., *Outlines of Industrial Chemistry*, 121.
Todd, D. P., Preparation of reticles, 59.
Trevor, J. E., Lachman's *Spirit of Organic Chemistry*, 120.
Waddell's *Arithmetic of Chemistry*, 120.
Van Deventer's *Physical Chemistry for Beginners*, 120.
Trowbridge, A., Theory of the coherer, 22.

U.

Ultra-violet light, development of kathode rays by, *Merritt and Stewart*, 230.
Units, the M. K. S. absolute system of, *Rogers*, 115.
Van der Waal's equation, Gibb's thermodynamic model for substance following, Boynton, 291.
Voltaic cell, thermodynamics of, *Carhart*, 1.

W.

Waddell, J., *Arithmetic of Chemistry*, 120.
Water, surface tension of, above 100° C., *Knipp*, 129.
Wave-length, effect on the persistence of vision, *Allen*, 257.
Williams, N. H., Verification of Boyle's law, 255.
Yeaman, *Central Station Electricity Supply*, 123.
Young, J. E., *Electrical Testing for Telegraph Engineers*, 64.

Ziegler Hand Dynamo.

The dynamo shown gives a current of four ampères, at an electromotive force of fifty volts. It will therefore permit of the employment of the ordinary 50-volt lamps of commerce for demonstration or experiment. It runs easily and is an ideal machine for the college and laboratory.

ZIEGLER ELECTRIC CO., 141 Franklin St., BOSTON, MASS.

Send for a copy of Physical Catalogue No. 10.

A History of Physics

In Its Elementary Branches, including the Evolution of Physical Laboratories

By FLORIAN CAJORI, Ph.D.

Professor of Physics in Colorado College.

Cloth Cr. 8vo, $1.60 *net.*

This brief popular History gives in broad outline the development of the science of physics from antiquity to the present time. It contains also a more complete statement than is found elsewhere of the evolution of physical laboratories in Europe and America. The book, while of interest to the general reader, is primarily intended for students and teachers of physics. The conviction is growing that, by a judicious introduction of historical matter, a science can be made more attractive. Moreover, the general view of the development of the human intellect which the history of a science affords is in itself stimulating and liberalizing.

"Prof. Cajori has followed his 'History of Mathematics' with a 'History of Physics' that is *even more interesting and valuable.* . . . Every teacher and student of physics should read it and add it to his library. It shows careful study, enthusiasm and a comprehensive grasp of the subject."—*Journal of Education.*

"A most valuable contribution to the subject."—*Scientific American.*

"Probably this will prove the most successful of all Prof. Cajori's histories."
—*The Nation.*

THE MACMILLAN COMPANY, Publishers, New York

Standard Text-Books on General Physics

The Elements of Physics

FOR USE IN HIGH SCHOOLS.

By HENRY CREW, Ph.D., Northwestern University.

Second Edition, Revised, **Cloth. $1.10,** *net.*

Elements of Theoretical Physics

By DR. C. CHRISTIANSEN

Professor of Physics, University of Copenhagen

TRANSLATED INTO ENGLISH BY

W.F. MAGIE, Ph.D., Professor of Physics, Princeton University **Cloth. 8vo. $3.25, net.**

Problems and Questions in Physics

CHARLES P. MATTHEWS, M.E., AND JOHN SHEARER, B.S.

Purdue University *Cornell University*

8vo. Cloth. pp. 247 + 4. Price $1.60, net

An Intermediate Course of Practical Physics

By A. SCHUSTER, Ph.D., F.R.S., AND C. H. LEES, D.Sc.

Owens College, Manchester *Owens College, Manchester*

12mo. Cloth. pp. xv + 248. Price $1.10 net.

Lessons on Elementary Practical Physics

By BALFOUR STEWART, A.M., LL.D., F.R.S., AND W. W. HALDANE GEE

Vol. I. **General Physical Processes.** 12mo. $1.50, *net*

Vol. II. **Electricity and Magnetism.** $2.25, *net*

Vol. III. Part I. **Practical Acoustics.** $1.10, *net*

Part II. **Heat and Light.** *In Press*

Laboratory Manual of Physics and Applied Electricity

Arranged and Edited by EDWARD L. NICHOLS

Professor of Physics in Cornell University

IN TWO VOLUMES **Vol. I. Cloth. Price $3.00, net**

JUNIOR COURSE IN GENERAL PHYSICS. By ERNEST MERRITT and FREDERICK J. ROGERS. **Vol. II. Cloth. pp. 444. Price $3.25, net**

SENIOR COURSES AND OUTLINES OF ADVANCED WORK By GEORGE S. MOLER, FRÈDERICK BEDELL, HOMER S. HOTCHKISS, CHAS. P. MATTHEWS, and THE EDITOR.

A large proportion of the students for whom primarily this Manual is intended are preparing to become engineers, and special attention has been devoted to the needs of that class of readers.

THE MACMILLAN COMPANY New York: 66 Fifth Avenue

BOSTON CHICAGO SAN FRANCISCO

IMPORTANT PUBLICATIONS

HISTORY OF MODERN PHILOSOPHY IN FRANCE. With twenty-three Portraits of French Philosophers. By Professor L. LÉVY-BRUHL, Maître de Conférences in the Sorbonne, Paris. Handsomely bound. Pages, 500. Price, **$3.00** (12s.).

THE HISTORY OF THE DEVIL AND THE IDEA OF EVIL. FROM THE EARLIEST TIME TO THE PRESENT DAY. By DR. PAUL CARUS. Printed in two colors from large type on fine paper. Bound in cloth, illuminated with cover stamp from Doré. 500 8vo pages, with 311 illustrations in black and tint. Price, **$6.00** (30s.).

SCIENCE AND FAITH, OR MAN AS AN ANIMAL AND MAN AS A MEMBER OF SOCIETY. WITH A DISCUSSION OF ANIMAL SOCIETIES. By DR. PAUL TOPINARD, Late General Secretary of the Anthropological Society of Paris, and Sometime Professor in the School of Anthropology. Pp., 361. Price, cloth, gilt top, **$1.50** (6s. 6d.).

"A most interesting volume."—*Glasgow Herald.*

"Stimulating and suggestive."—*The Scotsman.*

THE EVOLUTION OF GENERAL IDEAS. By TH. RIBOT, Professor in the Collège de France. Authorized translation from the French by FRANCES A. WELBY. Pp., xi, 231. Price, cloth, **$1.25** (6s. 6d.).

"All that he writes is lucid and suggestive, and the course of lectures here translated is a characteristic contribution to psychology."—*Nature.*

ELEMENTARY ILLUSTRATIONS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS. By AUGUSTUS DE MORGAN. New reprint edition. With sub-headings and bibliography of English and foreign works on the Calculus. Red Cloth. Price, **$1.00** (4s. 6d.).

"The present work may be safely recommended to those students who are anxious to obtain a knowledge of the Calculus which shall be real and abiding."

—*The Speaker*, London.

"It aims not at helping students to cram for examinations, but to give a scientific explanation of the rationale of these branches of mathematics. Like all that De Morgan wrote, it is accurate, clear, and philosophic."—*Literary World.*

"It would be difficult to overestimate the value of De Morgan's works and of the importance of rigid accuracy upon which he everywhere insists."

—*The Speaker*, London.

A BRIEF HI TORY OF MATHEMATIC. An Authorized Translation of Dr. Karl Fink's *Geschichte der Elementar-Mathematik.* By WOOSTER WOODRUFF BEMAN and DAVID EUGENE SMITH. With biographical notes and full index. Pages, 333+xii. Cloth, **$1.50** (5s. 6d.).

"Dr. Fink's work is the most systematic attempt yet made to present a compendious history of mathematics."—*The Outlook.*

THE OPEN COURT PUBLISHING COMPANY

324 Dearborn Street, Chicago, Ill

LONDON: Kegan Paul, Trench, Trübner & Co., Ltd.

The Astrophysical Journal

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

EDITED BY

GEORGE E. HALE
Director of the Yerkes Observatory

AND

JAMES E. KEELER
Director of the Lick Observatory

WITH THE COÖPERATION OF

THE LEADING ASTROPHYSICISTS OF THE UNITED STATES AND EUROPE

Contents of Recent Numbers

OCTOBER 1899.

The Effect of Certain Impurities on the Spectra of Some Gases . PERCIVAL LEWIS
On Professor Keeler's Photometric Researches, by Photographic Methods, on the Nebula in Orion . . . . J. SCHEINER
Note on the Foregoing Article by Professor Scheiner . . . . JAMES E. KEELER
The Position of the Stars of Type IV and of the Variable Stars of Type III in Reference to the Milky Way . . . . . T. E. ESPIN.
Observations of Comet Spectra W. H. WRIGHT.
The Spectroscopic Binary Capella . W. W. CAMPBELL
The Variable Velocities in the Line of Sight of ϵ Librae, h Draconis, λ Andromedae, ϵ Ursae Minoris and ω Draconis W. W. CAMPBELL.
The Variable Velocity of α Ursae Minoris in the Line of Sight W. W. CAMPBELL
The Variable Velocity of Polaris EDWIN B. FROST
The Wave-Length of the Green Coronal Line, and Other Data Resulting from an Attempt to Determine the Law of Rotation of the Solar Corona W. W. CAMPBELL
The Ring Nebula in Lyra JAMES E. KEELER
Pressure in the Electric Spark JOHN FRED MOHLER
Minor Contributions and Notes

NOVEMBER 1899.

On the Relative Brightness of the Planets Mars and Jupiter, from Measurements with a New Photometer J. HARTMANN
The Variable Velocities of β Capricorni and υ Sagittarii in the Line of Sight W. W. CAMPBELL
Suggestions for the Determination of Stellar Parallax by Means of Photography FRANK SCHLESINGER
The Distribution of Stars in the Cluster Messier 13, in Hercules H. K. PALMER
The Periods of the Variable Stars in the Cluster Messier 5 . . S. I. BAILEY
Note on the Relation Between the Visual and Photographic Light Curves of Variable Stars of Short Period S. I. BAILEY
The Annular Nebula *H IV.* 13 in Cygnus JAMES E. KEELER
On the Effect of Pressure Upon the Wave-Lengths of the Lines of the Hydragen Spectrum . . . . . J. WILSING
On the Presence of Oxygen in the Atmospheres of Certain Fixed Stars DAVID GILL
Minor Contributions and Notes

DECEMBER 1899.

Robert Wilhelm Bunsen . . HENRY CREW
The Wave-Length of the Corona Line C. A. YOUNG
Density of Close Double Stars ALEXANDER ROBERTS
The Densities of the Variable Stars of the Algol-Type . . HENRY NORRIS RUSSELL
Note on the Spectrum of P Cygni A. BÉLOPOLSKY
Apparatus and Method for the Photographic Measurement of the Brightness of Surfaces J. HARTMAN
The Great Sun-Spot of September 1898 J. FÉNYI
A Spectroscope of Fixed Deviation PH. PELLIN AND ANDRÉ BROCA
Researches on the Arc-Spectra of the Metals B. HASSELBERG
Reviews

JANUARY 1900.

The Spiral Nebula *H. I, 55 PEGASI* JAMES E. KEELER
On a New System for Spectral Photometric Work . . . . . D. B. BRACE
Calibration of the Slit in Spectral Photometric Measurements . . E. V. CAPPS
On the Escape of Gases from Planetary Atmospheres According to the Kinetic Theory . . . . . S. R. COOK
On a Balancing Rheostat for Bolometers P. G. NUTTING
Solar Eclipse Problems GEORGE E. HALE
Researches on the Arc-Spectra of the Metals B. HASSELBERG
On the Deviations from the Law of Reciprocity for Bromide of Silver Gelatine K. SCHWARZSCHILD
On the Effect of Intermittent Exposure on Bromide of Silver Gelatine K. SCHWARZSCHILD
Minor Contributions and Notes

CHICAGO

The University of Chicago Press

LONDON: WM WESLEY & SON

Published every month except July and September

Subscription Price
Domestic, $4.00; Foreign, $4.50